Appendix:

The Macroeconomics of Sticky Prices with Generalized Hazard Functions

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A Proofs of the Theorems

Proof. (of Lemma 1). Define the function $U(x) \equiv v(x) - v(0)$ and rewrite equation (3) as

$$rU(x) = Bx^2 + \frac{\sigma^2}{2}(U''(x) - v''(0)) - \kappa \int_0^{U(x)} G(\psi) d\psi \qquad \text{for } x \in [0, X] \tag{50}$$

with boundary conditions $U'(X) = 0$ and $U(X) = \Psi$. Note that by definition $U(0) = 0$. To obtain equation (50) we used integration by parts on the right hand side of equation (3):

$$\begin{aligned} \int_0^{U(x)} [\psi - U(x)] G'(\psi) d\psi &= G(\psi)\psi \Big|_0^{U(x)} - \int_0^{U(x)} G(\psi) d\psi - U(x) \int_0^{U(x)} G'(\psi) d\psi \\ &= G(\psi)\psi \Big|_0^{U(x)} - \int_0^{U(x)} G(\psi) d\psi - U(x) \left[G(U(x)) - G(0) \right] \\ &= -\int_0^{U(x)} G(\psi) d\psi + U(x) G(0) \end{aligned}$$

Next differentiate both sides of equation (50) with respect to x to obtain:

$$\left[r + \kappa G(U(x)) \right] U'(x) = 2Bx + \frac{\sigma^2}{2} U'''(x) \;\; \text{for } x \in [0, X] \tag{51}$$

with boundary conditions given by: $U'(X) = 0$ and $U'(0) = 0$. The first boundary condition is smooth pasting. Note that if $X = \infty$ we do not have smooth pasting, but since v is bounded above so is U, then it must be that $\lim_{x\to\infty} U'(x) = 0$, and hence the analogous boundary condition holds in the case where X is unbounded. The second boundary is implied by the symmetry and differentiability of $v(\cdot)$, and hence of $U(\cdot)$, around $x = 0$. Thus, solving for the value function in equation (3) is equivalent to solving for $U(\cdot)$ in equation (51) with its corresponding boundary conditions.

Now define $u(x) \equiv U'(x)$ and rewrite equation (51) using that $\Lambda(x) = \kappa G(U(x))$, by equation (2). This gives the o.d.e. in equation (4). The boundary conditions described above in terms of U' thus become $u(X) = u(0) = 0$.

Uniqueness and invertibility. Note that equation (4) is a linear second order ordinary differential equation of the Sturm-Liouville type with two Dirichlet boundary conditions, where we write: $L(u)(x) \equiv [r + \Lambda(x)]\, u(x) - \frac{\sigma^2}{2} u''(x)$ and thus the equation above can be written as $L(u)(x) = 2Bx$.

The function $\Lambda(\cdot)$ defining the operator L is continuous, so it has a unique solution $u(\cdot)$. To see this let $L(u)(x) = 2Bx$ and let $\{\theta_j, \varphi_j\}$ be the eigenvalues and orthonormal eigenfunctions of L satisfying the Dirichlet boundary conditions, i.e. solving $L(\varphi_j) = \theta_j \varphi_j$ and with $\varphi_j(0) = \varphi_j(X) = 0$. By linearity we have $L(\sum_j \alpha_j \varphi_j) = \sum_j \theta_j \alpha_j \varphi_j$ for any square integrable sequence $\{\varphi_j(\cdot)\}$. Then we can choose $\{\alpha_j\}$ so that $u(x) = \sum_j \theta_j \alpha_j \varphi_j(x)$, with the equality in the $\mathbf{L}^2$ sense. In particular we can set $\alpha_j = \langle \varphi_j, u \rangle / \theta_j$. Again, the case of $X = \infty$, requires a slightly different argument for the existence of its solution. In particular, the existence of a solution is guaranteed by Theorem 3.1 in Lian, Wang, and Ge (2009). By the Maximum principle then $u(x) > 0$ since $2Bx > 0$ in $(0, X)$. Since $u > 0$ then U is increasing and thus it is invertible.

Value function. We construct $v(\cdot)$ as follows. Recall $u = U'$ and $U(0) = 0$, we have

$$U(x) = \int_0^x u(z)dz \text{ for all } x \in [0, X] \quad \text{and} \quad \Psi = U(X) \; .$$

From the definition of $U(x) = v(x) - v(0)$ and equation (3) we have

$$v''(0) = U''(0) = u'(0) \text{ and } rv(0) = v''(0)\frac{\sigma^2}{2} \text{ so } v(0) = u'(0)\frac{\sigma^2}{2r}$$

which gives equation (5) in the lemma. Note that $v(\cdot)$ is increasing because $u(x) > 0$ on $(0, X)$ as established above. □

Proof. (of Theorem 1). We now construct the fixed cost Ψ, the Poisson arrival rate κ, the value of $G(0)$ and the density $G'(\cdot)$ that rationalize the generalized hazard rate $\Lambda(\cdot)$ using the function $u(\cdot)$. We use equation (2), $\Lambda(x) = \kappa G(U(x))$ for all $x \in [0, X]$, which evaluated at $x = 0$ implies $\Lambda(0) = \kappa G(0)$. Denote by $w(\cdot) \equiv U^{-1}(\cdot)$, the inverse function of $U(\cdot)$, mapping $[0, \Psi]$ onto $[0, X]$. Set κ to be $\kappa = \Lambda(X)$ to ensure that $G(\Psi) = 1$. Differentiating the expression above with respect to x, we have $G'(U(x))\, U'(x) = \frac{\Lambda'(x)}{\Lambda(X)}$ for all $x \in (0, X)$ and thus

$$G'(\psi) = G'(U(w(\psi))) = \frac{\Lambda'(w(\psi))}{u(w(\psi))\Lambda(X)} = \frac{\Lambda'(w(\psi))}{u(w(\psi))\Lambda(X)} \text{ for all } \psi \in (0, \Psi)$$

which gives the density of G' in terms of the function u defined in Lemma 1. □

Proof. (of Theorem 2) Without loss of generality, given the assumed symmetry, let $q(\cdot)$ be the density of minus price changes, so that $q(x)N_a = \Lambda(x)f(x)$. Denote the minus price changes by Δp. We will use four equations for $x > 0$:

$$\begin{aligned} f''(x) &= \frac{2}{\sigma^2} q(x) N_a \\ f'(x) &= f'(X) - \int_x^X f''(t)dt \\ f(x) &= -\int_x^X f'(t)dt \\ \sigma^2 &= N_a Var(\Delta p) \end{aligned}$$

where we have used that $f(X)=0$. Combining the first and the second equation we have ,

$$f'(x) = f'(X) - \frac{2}{\sigma^2} N_a \int_x^X q(x)dx = f'(X) - \frac{2}{\sigma^2} N_a \left(1 + f'(X)\frac{\sigma^2}{2N_a} - Q(x)\right)$$
$$= \frac{2}{\sigma^2} N_a (Q(x) - 1)$$

where we have used that $\lim Q(x) \to 1 + f'(X)\frac{\sigma^2}{2N_a}$ as $x \to X$. Integrating further,

$$f(x) = \frac{2}{\sigma^2} N_a \int_x^\infty (1 - Q(t))dt$$

Now using the last equation,

$$f(x) = \frac{2}{Var(\Delta p)} \int_x^\infty (1 - Q(t))dt$$

Using the identity $q(x)N_a = \Lambda(x)\bar{p}(x)$ once again,

$$\Lambda(x) = \frac{N_a Var(\Delta p)}{2} \frac{q(x)}{\int_x^\infty (1 - Q(t))dt}$$

Finally, we check whether $\Lambda(X) = \kappa < \infty$. If $X < \infty$, then using L'Hopital we get

$$\Lambda(X) = \frac{N_a Var(\Delta p)}{2} \frac{q'(X)}{-f'(X)\frac{\sigma^2}{2}} < \infty$$

If $X = \infty$, we apply L'Hopital rule twice, since $q(x) \to 0$ and $Q(x) \to 1$ as $x \to \infty$. We obtain:

$$\Lambda(X) = \frac{N_a Var(\Delta p)}{2} \lim_{x\to\infty} \frac{q''(x)}{q(x)}$$

which is finite given our assumption on the tail of q. This completes the proof. $\square$

Proof. (of Theorem 3). First note that the identity in equation (13), $N \cdot Var = \sigma^2$, holds in the model. Let $x(0)=0$. Consider the process $z(t) \equiv x(t)^2 - \sigma^2 t$ for $t \geq 0$. Using Ito's lemma we can verify that the drift of x^2 is σ^2, and hence $z(t)$ is a Martingale. Let τ be a stopping time, i.e. an instant where a price adjustment occurs (anywhere in the state space, including the boundaries), so that x is reset at $x(0)=0$. By the optional sampling theorem $z(\tau)$, the process stopped at τ, is also a martingale. Then $\mathbb{E}\left[z(\tau) \;\middle|\; x(0)\right] = \mathbb{E}\left[x(\tau)^2 \;\middle|\; x(0)\right] - \sigma^2 \mathbb{E}\left[\tau \;\middle|\; x(0)\right] = x(0) = 0$. Since $N = 1/\mathbb{E}\left[\tau \;\middle|\; x(0)\right]$ and $Var = \mathbb{E}\left[x(\tau)^2 \;\middle|\; x(0)\right]$ we get the identity in equation (13).

For simplicity, we focus next on the case with unbounded support $\bar{X} \to \infty$ (the logic for the case with bounded support is identical but the equations are slightly more cumbersome). Using the definition of the density of price changes in equation (15) we can rewrite the identity as

$$\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx = \sigma^2 \tag{52}$$

it is then straightforward to write the formula for kurtosis over $6N_a$ as:

$$\frac{Kur}{6N_a} = \frac{\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx}{6 \left(\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx \right)^2} = \frac{\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx}{6\sigma^4}$$

where the last passage uses equation (52). Using the Kolmogorov forward equation,

$$\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx = \frac{\sigma^2}{2} \int_{-\infty}^{\infty} x^4 f''(x) dx$$

Integrating by parts twice gives

$$\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx = 6\sigma^2 \int_{-\infty}^{\infty} x^2 f(x) dx$$

This allows us to write

$$\frac{Kur}{6N_a} = \frac{\int_{-\infty}^{\infty} x^2 f(x) dx}{\sigma^2} \tag{53}$$

Recall that we have a system of two equations:

$$\Lambda(x) f(x) = \frac{\sigma^2}{2} f''(x) \quad , \quad \Lambda(x) m(x) = \frac{\sigma^2}{2} m''(x) - x$$

Eliminate Λ to get:

$$\frac{\sigma^2}{2} \frac{m(x) f''(x)}{f(x)} = -x + \frac{\sigma^2}{2} m''(x)$$

Multiply both sides by $f(x)x$ and rearrange:

$$\frac{\sigma^2}{2} [m(x) f''(x) - m''(x) f(x)] x = -x^2 f(x)$$

Integrate both sides from 0 to ∞:

$$\frac{\sigma^2}{2} \int_0^{\infty} [m(x) f''(x) - m''(x) f(x)] x dx = -\int_0^{\infty} x^2 f(x) dx$$

Perform integration by parts in the left-hand side using the fact that $[m(x)f'(x) - m'(x)f(x)]' = m(x)f''(x) - m''(x)f(x)$:

$$\begin{aligned} \frac{\sigma^2}{2} \int_0^{\infty} [m(x) f''(x) - m''(x) f(x)] x dx &= \frac{\sigma^2}{2} \left([m(x) f'(x) - m'(x) f(x)] x \Big|_0^{\infty} - \int_0^{\infty} [m(x) f'(x) - m'(x) f(x)] dx \right) \\ &= -\sigma^2 \int_0^{\infty} m(x) f'(x) dx \end{aligned}$$

where the last equality uses integration by parts again. We used $\mathbb{E}[m(x)] < \infty$ and $m(\cdot)$ being almost linear at infinity to justify setting $f'(x)m(x)x$ and $f(x)m'(x)x$ at infinity to 0. Hence, we have

$$\sigma^2 \int_0^\infty m(x)f'(x)dx = \int_0^\infty x^2 f(x)dx$$

Plugging this result in equation (53) we have

$$\frac{Kur}{6N_a} = \int_{-\infty}^{\infty} m(x)f'(x)dx$$

It follows from the definition of $\mathcal{M}(\delta)$ in equation (33) that the right hand side is the first derivative of the CIR with respect to δ, evaluated at $\delta = 0$, or $\mathcal{M}'(\delta)$. This completes the proof. □

Proof. (of Theorem 4) The proof follows the same steps used for Theorem 1.

Proof. (of Theorem 5) The proof will proceed in three steps.

Step 1. As a first step we show how to recover the value μ and σ^2 based on the distribution of Δp. The first moment of Δp, together with the frequency N gives μ from $\mathbb{E}[\Delta p] = -\mu N_a$. To see this, let τ the stopping time at which prices are changed, so we have $\Delta p = x^* - x(\tau)$ with $x(0) = x^*$ and $dx = \mu dt + \sigma dW$. Let $y(t) = x^* - x(t) + \mu t$. Note that y is a Martingale, and thus $\mathbb{E}[y(\tau)] = 0 = \mathbb{E}[x^* - x(\tau)] + \mu\mathbb{E}[\tau]$ or $N_a\mathbb{E}[\Delta p] = -\mu$. To obtain σ^2 we prove that the moment generating function of the distribution of price changes must equal one when evaluated at $2\mu/\sigma^2$, i.e.:

$$M_{\Delta p}\left(\frac{2\mu}{\sigma^2}\right) = \int e^{\frac{2\mu}{\sigma^2}\Delta p} dQ(\Delta p) = 1$$

where $M_{\Delta p}(\cdot)$ is the moment generating function of the distribution of prices. To see why this has to be the case, we first define F_n as:

$$F_n \equiv \frac{1}{N_a}\left[\int_{\underline{x}}^{\overline{x}} \tfrac{\sigma^2}{2} f''(x)(x^* - x)^n dx - \tfrac{\sigma^2}{2} f'(x)(x^* - x)^n\Big|_{\underline{x}}^{\overline{x}}\right]$$

Using the KFE equation, we get

$$\begin{aligned} F_n &= \frac{1}{N_a}\int_{\underline{x}}^{\overline{x}} \left[\Lambda(x)f(x) + \mu f'(x)\right](x^* - x)^n dx - \frac{\sigma^2}{2N_a} f'(x)(x^* - x)^n\Big|_{\underline{x}}^{\overline{x}} \\ &= \int_{\underline{x}}^{\overline{x}} (x^* - x)^n q(x^* - x)dx - \frac{\sigma^2}{2N_a} f'(x)(x^* - x)^n\Big|_{\underline{x}}^{\overline{x}} + \frac{1}{N_a}\mu\int_{\underline{x}}^{\overline{x}} f'(x)(x^* - x)^n dx \end{aligned}$$

Using the definition of q and integrating by parts:

$$\begin{aligned} F_n = &\int_{\underline{x}}^{\overline{x}} (x^* - x)^n q(x^* - x)dx - \frac{\sigma^2}{2N_a} f'(x)(x^* - x)^n\Big|_{\underline{x}}^{\overline{x}} \\ &+ \frac{2\mu}{\sigma^2}\frac{1}{n+1}\frac{1}{N_a}\left[\int_{\underline{x}}^{\overline{x}} \tfrac{\sigma^2}{2} f''(x)(x^* - x)^{n+1} dx - \tfrac{\sigma^2}{2} f'(x)(x^* - x)^{n+1}\Big|_{\underline{x}}^{\overline{x}}\right] \end{aligned}$$

This implies

$$F_n = \mathbb{E}\left[\Delta p^n\right] + \frac{2\mu}{\sigma^2}\frac{1}{n+1}F_{n+1}$$

with

$$\begin{aligned}
F_1 &= \frac{1}{N_a}\left[\int_{\underline{x}}^{\overline{x}} \frac{\sigma^2}{2}(x^*-x)f''(x)dx - \frac{\sigma^2}{2}f'(x)(x^*-x)|_{\underline{x}}^{\overline{x}}\right] \\
&= \frac{1}{N_a}\left[\int_{\underline{x}}^{\overline{x}} \frac{\sigma^2}{2}f'(x)dx + \frac{\sigma^2}{2}(x^*-x)f'(x)|_{\underline{x}}^{\overline{x}} - \frac{\sigma^2}{2}f'(x)(x^*-x)|_{\underline{x}}^{\overline{x}}\right] \\
&= \frac{\sigma^2}{2N_a}f(x)|_{\underline{x}}^{\overline{x}} = 0
\end{aligned}$$

Iterating the recursive expression for F_n with this starting condition and assuming that $\mu \neq 0$,

$$0 = \frac{2\mu}{\sigma^2}F_1 = \sum_{j=1}^{\infty}\left(\frac{2\mu}{\sigma^2}\right)^j \frac{1}{j!}\mathbb{E}\left[(\Delta p)^j\right]$$

Now the moment generating function can be written as

$$M_{\Delta p}(\varphi) \equiv \int e^{\varphi\Delta p}dQ(\Delta p) = 1 + \sum_{j=1}^{\infty}\frac{\varphi^j}{j!}\int(\Delta p)^j dQ(\Delta p)$$

Hence $M_{\Delta p}(\frac{2\mu}{\sigma^2}) = 1$.

Step 2. As an intermediate step we develop an alternative expression for N_a, which will be used below. By a mass preservation argument in the time dependent version of the Kolmogorov equation, continuity of f at $x = x^*$, and the boundary conditions at $x = \overline{x}$ and $x = \underline{x}$,

$$0 = \frac{\sigma^2}{2}[f'(\overline{x}) - f'(x^*_+) + f'(x^*_-) - f'(\underline{x})] - \int_{\underline{x}}^{\overline{x}}\Lambda(x)f(x)dx$$

Replacing the expression for N_a we obtain $N_a = \frac{\sigma^2}{2}[f'(x^*_-) - f'(x^*_+)]$.

Step 3. Now we turn to obtain the invariant distribution of of price gaps f. Using the definition of the density of price changes into the Kolmogorov forward equation for any $x \in (\underline{x}, \overline{x})/\{x^*\}$,

$$f''(x) = \frac{2}{\sigma^2}\left[f'(x)\mu + \Lambda(x)f(x)\right] = \frac{2\mu}{\sigma^2}f'(x) + \frac{2N_a}{\sigma^2}q(x^*-x)$$

which is a non-homogenous first order ordinary differential equation with constant coefficient. Letting $\overline{a} \equiv f'(x^*_+) < 0$, and $\underline{a} \equiv f'(x^*_-) > 0$, we can solve the initial value problem for f' using the definition of the function R in the statement of the theorem:

$$\begin{aligned}
f'(x;\underline{a}) &= e^{\frac{2\mu}{\sigma^2}(x-x^*)}\left[\underline{a} - \frac{2N_a}{\sigma^2}R\left(0, x^*-x\right)\right] \text{ for } x \in (\underline{x}, x^*] \\
f'(x;\overline{a}) &= e^{\frac{2\mu}{\sigma^2}(x-x^*)}\left[\overline{a} + \frac{2N_a}{\sigma^2}R\left(x^*-x, 0\right)\right] \text{ for } x \in [x^*, \overline{x})
\end{aligned}$$

which can be verified to solve the first order linear o.d.e. for f' in each of the segments. Hence,

$$f(x) = f(x^*) + \frac{2N_a}{\sigma^2}\int_x^{x^*} e^{\frac{2\mu}{\sigma^2}(z-x^*)}R(0, x^*-z)dz + \frac{\underline{a}\sigma^2}{2\mu}\left[e^{\frac{2\mu}{\sigma^2}(x-x^*)}-1\right] \text{ for } x \in (\underline{x}, x^*]$$
$$f(x) = f(x^*) + \frac{2N_a}{\sigma^2}\int_{x^*}^{x} e^{\frac{2\mu}{\sigma^2}(z-x^*)}R(x^*-z, 0)dz + \frac{\overline{a}\sigma^2}{2\mu}\left[e^{\frac{2\mu}{\sigma^2}(x-x^*)}-1\right] \text{ for } x \in [x^*, \overline{x})$$

We now derive two equations that $\underline{a}$ and $\overline{a}$ must satisfy. Imposing $f(\overline{x}) = f(\underline{x}) = 0$ and that f is continuous at $x = x^*$, we get

$$f(x^*) = \int_{\underline{x}}^{x^*} f'(x;\underline{a})dx = -\frac{2N_a}{\sigma^2}\int_{\underline{x}}^{x^*} e^{\frac{2\mu}{\sigma^2}(x-x^*)}R\left(0, x^*-x\right)dx + \frac{\underline{a}\sigma^2}{2\mu}\left[1-e^{\frac{2\mu}{\sigma^2}(\underline{x}-x^*)}\right]$$
$$= -\int_{x^*}^{\overline{x}} f'(x;\overline{a})dx = -\frac{2N_a}{\sigma^2}\int_{x^*}^{\overline{x}} e^{\frac{2\mu}{\sigma^2}(x-x^*)}R\left(x^*-x, 0\right)dx - \frac{\overline{a}\sigma^2}{2\mu}\left[e^{\frac{2\mu}{\sigma^2}(\overline{x}-x^*)}-1\right]$$

Thus, the system of two linear independent equations in $\underline{a}$ and $\overline{a}$ is:

$$\frac{2N_a}{\sigma^2}\left[\int_{\underline{x}}^{x^*} e^{\frac{2\mu}{\sigma^2}(x-x^*)}R\left(0, x^*-x\right)dx - \int_{x^*}^{\overline{x}} e^{\frac{2\mu}{\sigma^2}(x-x^*)}R\left(x^*-x, 0\right)dx\right]$$
$$= \underline{a}\left[\frac{1-e^{\frac{2\mu}{\sigma^2}(\underline{x}-x^*)}}{2\mu/\sigma^2}\right] + \overline{a}\left[\frac{e^{\frac{2\mu}{\sigma^2}(\overline{x}-x^*)}-1}{2\mu/\sigma^2}\right]$$
$$\frac{2N_a}{\sigma^2} = \underline{a} - \overline{a}$$

To arrive at the expressions in the statement of the theorem, replace $2\mu/\sigma^2$ with ϕ and normalize $\overline{a}$ and $\underline{a}$ by $2N_a/\sigma^2$:

$$\left[\int_{\underline{x}}^{x^*} e^{\phi(x-x^*)}R\left(0, x^*-x\right)dx - \int_{x^*}^{\overline{x}} e^{\phi(x-x^*)}R\left(x^*-x, 0\right)dx\right] = \underline{a}\left[\frac{1-e^{\phi(\underline{x}-x^*)}}{\phi}\right] + \overline{a}\left[\frac{e^{\phi(\overline{x}-x^*)}-1}{\phi}\right]$$
$$1 = \underline{a} - \overline{a}$$

The expression for $f(x^*)$ is now

$$f(x^*) = -\frac{2N_a}{\sigma^2}\left(\int_{x^*}^{\overline{x}} e^{\phi(x-x^*)}R\left(x^*-x, 0\right)dx - \frac{\overline{a}}{\phi}\left[e^{\phi(\overline{x}-x^*)}-1\right]\right)$$

The expressions for $f(x)$ on both sides of x^* are

$$f(x) = f(x^*) + \frac{2N_a}{\sigma^2}\left(\int_x^{x^*} e^{\phi(z-x^*)}R(0, x^*-z)dz + \frac{\underline{a}}{\phi}\left[e^{\phi(x-x^*)}-1\right]\right) \text{ for } x \in (\underline{x}, x^*]$$
$$f(x) = f(x^*) + \frac{2N_a}{\sigma^2}\left(\int_{x^*}^{x} e^{\phi(z-x^*)}R(x^*-z, 0)dz + \frac{\overline{a}}{\phi}\left[e^{\phi(x-x^*)}-1\right]\right) \text{ for } x \in [x^*, \overline{x})$$

It is now straightforward to change the variables from x to $y = x - x^*$ and go from $f(x)$ to $\tilde{f}(y + x^*) = f(x)/(2N_a/\sigma^2)$. This completes the proof. □

B Proofs of the Propositions

Proof. (of Proposition 1) The details are in Section G. The proof follows the same steps as that of Theorem 1.

Proof. (of Proposition 2) To show this let the density of the invariant distribution be $\tilde{f}(z) = f(z/b)/b$. This function solves the KFE for $\tilde{\Lambda}$ and $\tilde{\sigma}^2$. This can be verified using that f solves the KFE for Λ and σ^2. Since $N_a = -\sigma^2 f'(0)$ and $\tilde{N}_a = -\tilde{\sigma}^2 \tilde{f}'(0)$ then it implies that $\tilde{N}_a = N_a$ for any b. Also we can see that $\tilde{q}(z) = q(z/b)/b$, by using $q(x) = \Lambda(x) f(x)/N_a$ and $\tilde{q}(z) = \tilde{\Lambda}(z)\tilde{f}(z)/\tilde{N}_a$ for all $z \in (-X\,b, X\,b)$. Using the formula for a change on variable, and the relationship between q and $\tilde{q}$ and of Λ and $\tilde{\Lambda}$ we get $\int_{-X}^{X} \Lambda(x) f(x) dx = \int_{-\tilde{X}}^{\tilde{X}} \tilde{\Lambda}(z)\tilde{f}(z) dz$, and thus $\tilde{s} = s$. $\square$

Proof. (of Proposition 3) We start by describing the o.d.e and boundary that f and f_k satisfy. For f we have:

$$\begin{aligned}
\Lambda(x) f(x) &= \frac{\sigma^2}{2} f''(x) \text{ for all } x \in (0, X) \\
f(X) &= 0 \\
1/2 &= \int_0^X f(x) dx
\end{aligned}$$

For f_k we have

$$\begin{aligned}
\Lambda(x) f_k(x) &= \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (0, X) \\
k f(x) &= \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (X, \infty) \\
1/2 &= \int_0^X f_k(x) dx + \int_X^\infty f_k(x) dx
\end{aligned}$$

and that p_k has a continuous first derivative at $x = X$. We can then solve for f_k for $x > X$, obtaining $f_k(x) = f_k(X) e^{-\eta(x-X)}$ for all $x > X$, where $\eta = \sqrt{2k}/\sigma$. Thus, using the required continuity we can write:

$$\begin{aligned}
\Lambda(x) f_k(x) &= \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (0, X) \\
f_k'(X) &= -\eta f_k(X) \\
1/2 &= \int_0^X f_k(x) dx + f_k(X)/\eta
\end{aligned}$$

Now consider the solutions of the homogenous second order o.d.e. given by $\sigma^2/2 f''(x) = \Lambda(x) f(x)$ for $x \in [0, X]$. Given the assumption that Λ is continuous, we know that the solution is given by linear combinations of two linearly independent functions g_1, g_2 defined $[0, X]$. This functions depend on the interval $(0, X)$, the constant $\sigma > 0$ only. Thus we can write the solution of each of the two o.d.e. above as:

$$\begin{aligned}
f_k(x) &= a_k g_1(x) + b_k g_2(x) \\
f(x) &= a g_1(x) + b g_2(x)
\end{aligned}$$

for all $x \in [0, X]$. The coefficients a_k, b_k, a, b can be chosen to satisfy the two boundary conditions written for f and f_k. We can use the homogeneity of the boundary conditions and preliminary set $a_k = a = 1$, drop the boundary conditions given by the integral equation for each system, use $\bar{b}, b_k$ to solve the remaining boundary conditions at X, and then find a, a_k and rescale b, b_k to satisfy the two integral equations. To do so, let $\hat{b} = b/a$ and $\hat{b}_k = a_k/b_k$. Thus we write the remaining boundary conditions:

$$
\begin{aligned}
f(X) = 0 &\text{ becomes } 0 = g_1(X) + \hat{b}g_2(X) \\
f_k'(X) = -\eta f_k(X) &\text{ becomes } g_1'(X) + \hat{b}_k g_2'(X) = -\eta \left[g_1(X) + \hat{b}_k g_2(X)\right]
\end{aligned}
$$

equivalently we can write:

$$
\hat{b} = -\frac{g_1(X)}{g_2(X)} \text{ and } \hat{b}_k = -\frac{\eta g_1(X) + g_1'(X)}{\eta g_2(X) + g_2'(X)}
$$

Furthermore let $I_i \equiv \int_0^X g_i(x)dx$ for $i = 1, 2$ so that we can write the remaining boundary conditions as:

$$
\begin{aligned}
&1/2 = aI_1 + bI_2 \implies a = \frac{1}{2\left(I_1 + I_2\hat{b}\right)} \\
&1/2 = a_k I_1 + b_k I_2 + a_k \frac{g_1(X)}{\eta} + b_k \frac{g_2(X)}{\eta} \implies a_k = \left(I_1 + \hat{b}_k I_2 + \frac{g_1(X)}{\eta} + \hat{b}_k \frac{g_2(X)}{\eta}\right)/2
\end{aligned}
$$

Note that, given the expression for η, taking $k \to \infty$ it is equivalent to take $\eta \to \infty$. Then, using L'Hopital in the second equation we obtain that $\hat{b}_k \to \hat{b}$, which them implies that $a_k \to a$ and finally $b_k \to b$. Now we can compare f_k and f to obtain:

$$
\begin{aligned}
|f_k(x) - f(x)| &= |(a_k - a)|g_1(x) + (b_k - b)g_2(x)| \\
&\leq |a_k - a||g_1(x)| + |b_k - b||g_2(x)| \text{ for all } x \in [0, X]
\end{aligned}
$$

Since g_1 and g_2 are continuous in x, then they are bounded in $[0, X]$. Thus as $k \to \infty$ we have that f_k converges uniformly to f. $\square$

Proof. (of Proposition 4) Absolute continuity of $Q(\cdot)$ follows from continuity of $f(\cdot)$ on $(-X, X)/\{0\}$ and boundedness of $\Lambda(\cdot)$ on $(-X, X)$. Symmetry of $q(\cdot)$ follows from both $f(\cdot)$ and $\Lambda(\cdot)$ being symmetric, and its continuity follows from the continuity of $f(\cdot)$.

That $Q(\cdot)$ is fully identified by all its moment requires either $X < \infty$ or the existence of its moment generating function in some neighborhood of zero when $X = \infty$. This is Theorem 2.3.11 in Casella and Berger (2002). Take the case $X = \infty$. We will show the existence of the moment generating function in a neighborhood of zero, which amounts to convergence of a series

$$
\sum_{n=0}^{\infty} \frac{(ia)^n \mathbb{E}[x^n]}{n!}
$$

for some $a > 0$. Due to symmetry, all odd moments are zero, so we will prove that the even moments grow no faster than the factorial.

Consider an even moment $\mathbb{E}[x^{2k+2}]$:

$$\mathbb{E}\left[x^{2k+2}\right] = \int\limits_{-\infty}^{\infty} x^{2k+2} q(x) dx = \frac{2}{N_a} \int\limits_{0}^{\infty} x^{2k+2} \Lambda(x) f(x) dx = \frac{\sigma^2}{N_a} \int\limits_{0}^{\infty} x^{2k+2} f''(x) dx$$

This uses the definition of and symmetry $q(\cdot)$ and the KFE. Integrate the right-hand side by parts twice:

$$\frac{\sigma^2}{N_a} \int\limits_{0}^{\infty} x^{2k+2} f''(x) dx = \frac{\sigma^2 (2k+2)(2k+1)}{N_a} \int\limits_{0}^{\infty} x^{2k} f(x) dx$$

Here we used the fact that, due to Assumption 1, $\Lambda(\cdot)$ is bounded away from zero for $x > x^H$, so the decay rate of $q(\cdot)$ is no slower than exponential. This drives the intermediate terms from integration by parts to zero.

Now we will prove that

$$\int\limits_{0}^{\infty} x^{2k} f(x) dx \leq \xi \int\limits_{0}^{\infty} x^{2k} \Lambda(x) f(x) dx$$

for some number ξ that does not depend on k. Two cases are interesting. First is when there is a number $\lambda_1 > 0$ such that $\Lambda(x) > \lambda$ with probability one with respect to the measure defined by $f(\cdot)$. In this case,

$$\int\limits_{0}^{\infty} x^{2k} f(x) dx \int\limits_{0}^{\infty} \frac{1}{\Lambda(x)} x^{2k} \Lambda(x) f(x) dx < \frac{1}{\lambda} \int\limits_{0}^{\infty} x^{2k} \Lambda(x) f(x) dx$$

and we are done. Now assume, on the contrary, for any positive number λ there is a positive measure (corresponding to $f(\cdot)$) of x such that $\Lambda(x) < \lambda$. Recall that, by Assumption 1, there exist $x^H > 0$ and $\lambda > 0$ such that $\Lambda(x) > \lambda$ for $x > x^H$. The there exists a pair of numbers (λ_2, x_2) with and two sets A_1 and A_2 such that $A_1 = \{x : \Lambda(x) < \lambda_2\}$, $A_2 = [x_2, \infty)$, the measures of A_1 and A_2 associated with $f(\cdot)$ are equal to $F > 0$, and

$$\int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x) f(x) dx \right) f(x) dx = - \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x) f(x) dx \right) f(x) dx \qquad (54)$$

To see why these sets exist, take first $x_2 = x^H$. If there is no $\lambda_2 < \lambda$ such that the measure of $\{x : \Lambda(x) < \lambda_2\}$ is equal to $[x_2, \infty)$, increase x_2 until there is. Since $X = \infty$, the measure of $[x_2, \infty)$ decreases continuously as x_2 increases, so for any $\lambda_1 < \lambda$ the value of $x_2 \geq x^H$ such that the measures of A_2 and A_1 are equal exists.

Now consider the difference

$$F \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \int\limits_{A_1 \cup A_2} x^{2k}f(x)dx$$
$$= \int\limits_{A_1 \cup A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) x^{2k}f(x)dx$$
$$= \int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) x^{2k}f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) x^{2k}f(x)dx$$

Consider the last line. We know from equation (54) that the expression in brackets under the first integral is negative, and that under the second integral is positive. This is because they are the sum to zero, and $\Lambda(x)$ is greater on A_2 then on A_1. We also know that $x \le x^H$ on A_1 and $x \ge x^H$ on A_2. Hence,

$$\int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) x^{2k}f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) x^{2k}f(x)dx$$
$$\ge \left(x^H\right)^{2k} \left[\int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx\right) f(x)dx\right]$$
$$= 0 \tag{55}$$

This insures

$$\int\limits_{A_1 \cup A_2} x^{2k}f(x)dx \le \frac{F}{\int_{A_1 \cup A_2} \Lambda(x)f(x)dx} \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx = \xi_1 \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx$$

At the same time,

$$\int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}f(x)dx \le \frac{1}{\lambda_2} \int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}\Lambda(x)f(x)dx = \xi_2 \int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}\Lambda(x)f(x)dx$$

Hence,

$$\int\limits_0^\infty x^{2k}f(x)dx \le \max\{\xi_1, \xi_2\} \int\limits_0^\infty x^{2k}\Lambda(x)f(x)dx = \max\{\xi_1, \xi_2\}\mathbb{E}\left[x^{2k}\right]$$

Pluggin this to what was obtained before,

$$\mathbb{E}\left[x^{2k+2}\right] \le \frac{\sigma^2(2k+2)(2k+1)\max\{\xi_1, \xi_2\}}{N_a}\mathbb{E}\left[x^{2k}\right]$$

This implies that the series in question converges, and thus the moment generating function exists, at least in the circle of the radius $\sqrt{N_a/(\sigma^2 \max\{\xi_1, \xi_2\})}$. $\square$

Proof. (of Proposition 5) Under the identification assumptions,

$$\frac{\mathbb{E}[(\Delta p_{it})^j(\Delta p_{is})^k]}{\mathbb{E}[(\Delta p_{it})^{j'}(\Delta p_{is})^{k'}]} = \frac{\mathbb{E}[b_i^{j+k}(\Delta\tilde{p}_t)^j(\Delta\tilde{p}_s)^k]}{\mathbb{E}[b_i^{j'+k'}(\Delta\tilde{p}_t)^{j'}(\Delta\tilde{p}_s)^{k'}]} = \frac{\mathbb{E}[(b_i)^{j+k}]\mathbb{E}[(\Delta\tilde{p}_t)^j]\mathbb{E}[(\Delta\tilde{p}_s)^k]}{\mathbb{E}[(b_i)^{j'+k'}]\mathbb{E}[(\Delta\tilde{p}_t)^{j'}]\mathbb{E}[(\Delta\tilde{p}_s)^{k'}]}$$
$$= \frac{\mathbb{E}[(\Delta\tilde{p}_t)^j]\mathbb{E}[(\Delta\tilde{p}_t)^k]}{\mathbb{E}[(\Delta\tilde{p}_t)^{j'}]\mathbb{E}[(\Delta\tilde{p}_t)^{k'}]}$$

The first equality uses $\Delta p_{it} = b_i\Delta\tilde{p}_t$. The second one uses mutual independence of b_i, $\Delta\tilde{p}_t$, and $\Delta\tilde{p}_s$. The last one uses the fact that $\Delta\tilde{p}_t$ and $\Delta\tilde{p}_s$ are identically distributed. □

Proof. (of Proposition 6) Start with $Q(x)$:

$$Q(x) = \mathbb{P}\{\Delta p_{it} \le x\} = \int_0^\infty \mathbb{P}\left\{\Delta\tilde{p}_t \le \frac{x}{b_i}\Big| b_i\right\} dH(b_i) = \int_0^\infty \mathbb{P}\left\{\Delta\tilde{p}_t \le \frac{x}{b_i}\right\} dH(b_i) \tag{56}$$

The last equality uses the mutual independence of $\Delta\tilde{p}_t$ and b_i. Differentiate with respect to x:

$$q(x) = \partial_x\mathbb{P}\{\Delta p_{it} \le x\} = \int_0^\infty \frac{1}{b_i}\partial_x\mathbb{P}\left\{\Delta\tilde{p}_t \le \frac{x}{b_i}\right\} dH(b_i)$$

Evaluate at $x = 0$:

$$q(0) = \int_0^\infty \frac{1}{b_i}\tilde{q}(0)dH(b_i) = \mathbb{E}[b_i^{-1}]\ \tilde{q}(0) \tag{57}$$

Now turn to $\mathcal{C}_{pooled}$:

$$\mathcal{C}_{pooled} = \frac{q(0)}{2}\frac{Var(\Delta p_{it})}{\mathbb{E}[|\Delta p_{it}|]} = \frac{\tilde{q}(0)}{2}\frac{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}{\mathbb{E}[b_i]}\frac{Var(\Delta\tilde{p}_t)}{\mathbb{E}[|\Delta\tilde{p}_t|]} = \mathcal{C}\ \frac{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}{\mathbb{E}[b_i]}$$

Hence,

$$\mathcal{C} = \mathcal{C}\ \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \mathcal{C}_{pooled}\left(1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\right) < \mathcal{C}_{pooled}$$

That the correction multiplier is smaller then one follows from the correlation between $1/b_i$ and b_i^2 being negative. Next we find the expression for the correction as a function of the moments:

$$\frac{\mathbb{E}[|\Delta p_{it}|]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|]} = \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\frac{\mathbb{E}[|\Delta\tilde{p}_t|]}{\mathbb{E}[|\Delta\tilde{p}_t|^{-1}]\mathbb{E}[|\Delta\tilde{p}_t|^2]} = \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\frac{\mathbb{E}[|\Delta p_{it}|]}{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]}$$

Hence,

$$\frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \frac{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|]}$$

This completes the proof. □

Proof. (of Lemma 4) Denote $S^n(t) \equiv \frac{\partial^n}{\partial t^n} S(t)$. We will derive the following recursion:

$$S^{(n)}(t) = \mathbb{E}\left[F_n(x(t))e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \text{ for all } t \geq 0 \text{ and all } n = 1, 2, \ldots \tag{58}$$

for a sequence of functions $F_n : \mathbb{R} \to \mathbb{R}$. For $n = 1$ it follows from differentiating equation (76) with respect to t:

$$S^{(1)}(t) = -\mathbb{E}\left[\Lambda(x(t))e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \tag{59}$$

thus $F_1(x) = -\Lambda(x)$. For the induction step, assume that equation (58) hold and we will differentiate it with respect to t. To do this, since $F_n(x(t))$ is an Ito's process, and thus not differentiable with respect to time, we use Ito's lemma for the product of two Ito's process, namely $F_n(x(t))$ and $Z(t) \equiv e^{-\int_0^t \Lambda(x(s))ds}$, the second one being a degenerate one, since it has bounded variation. We then use that $dF_n(x(t)) = \partial_{xx}F_n(x(t))\frac{\sigma^2}{2}dt + \partial_x F_n(x(t)\sigma dW$, since x has no drift, and $dZ(t) = -\Lambda(x(t))Z(t)dt$. Thus,

$$\begin{aligned}
S^{(n+1)}(t) &\equiv \lim_{\Delta \downarrow 0} \frac{S^{(n)}(t+\Delta) - S^{(n)}(t)}{\Delta} \\
&= \lim_{\Delta \downarrow 0} \frac{1}{\Delta}\mathbb{E}\left[F_n(x(t+\Delta))Z(t+\Delta) - F_n(x(t))Z(t) \,|\, x(0) = 0\right] \\
&= \mathbb{E}\left[\left(\frac{\sigma^2}{2}\partial_{xx}F_n(x(t)) - \Lambda(x(t))F_n(x(t))\right) Z(t) \,|\, x(0) = 0\right] \\
&= \mathbb{E}\left[\left(\frac{\sigma^2}{2}\partial_{xx}F_n(x(t)) - \Lambda(x(t))F_n(x(t))\right) e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right]
\end{aligned}$$

which give us a recursion for F_n:

$$F_{n+1}(x) = \frac{\sigma^2}{2}\partial_{xx}F_n(x) - \Lambda(x)F_n(x) \text{ for all } x \tag{60}$$

Finally, evaluating the n^{th} derivatives of S at $t = 0$ we have:

$$S^{(n)}(0) = F_n(0) \text{ and all } n = 1, 2, \ldots \tag{61}$$

This completes the proof. □

Proof. (of Proposition 7) In the text. □

Proof. (of Proposition 8) Let the price gap distributions that correspond to Λ_1 and Λ_2 be f_1 and f_2. Recall that for a fixed N_a and σ^2 we have $f_1'(0) = f_2'(0)$ and it is sufficient to compare

$$\int_0^\infty f_1(x)x^2dx \text{ against } \int_0^\infty f_2(x)x^2dx$$

(1) We first claim that the graph of the function $f_1(x) - f_2(x)$ cannot cross the $x-$axis from above. That is, there is no segment $[a, b]$ such that $f_1(x) - f_2(x) = 0$ on this segment, $f_1(x) - f_2(x) > 0$ to the left of a, and $f_1(x) - f_2(x) > 0$ to the right of b. Note that this nests the case when $a = b$ and hence $[a, b]$ is a single point. Suppose such a segment exists. Then one of the two statements is true: either $\Lambda_1(x) \geq \Lambda_2(x)$ for all $x \leq a$ or $\Lambda_1(x) \leq \Lambda_2(x)$ for all $x \geq b$.

In the first case, the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the left of a. If it does cross it at some $c < a$, on (c, a) we have $f_1(x) > f_2(x)$ and hence $\Lambda_1(x)f_1(x) > \Lambda_2(x)f_2(x)$, implying $f_1''(x) > f_2''(x)$. But this contradicts $f_1'(c) - f_2'(c) \geq 0$ and $f_1'(a) - f_2'(a) \leq 0$ holding simultaneously. Hence, for all $x < a$ we have $f_1(x) > f_2(x)$, implying $\Lambda_1(x)f_1(x) > \Lambda_2(x)f_2(x)$ and $f_1''(x) > f_2''(x)$ on $(0, a)$. But since $f_1'(a) \leq f_2'(a)$, in this region we have $f_1'(x) < f_2'(x)$, which contradicts $f_1'(0) = f_2'(0)$.

In the second case, the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the right of b. If it does cross it at some $d > b$, on (b, d) we have $f_1(x) < f_2(x)$ and hence $\Lambda_1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$. But this contradicts $f_1'(b) - f_2'(b) \leq 0$ and $f_1'(d) - f_2'(d) \geq 0$ holding simultaneously. Hence the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the right of b, which already rules out $X_1 > X_2$. Moreover, if $X_1 = X_2 \leq \infty$, it must hold that $f_1'(X_1) \geq f_2'(X_1)$, which contradicted by $f_1'(x) < f_2'(x)$ for $x > b$. The latter follows from $f_1'(b) - f_2'(b) \leq 0$ and $f_1''(x) < f_2''(x)$ for $x > b$.

(2) Since the graph of the function $f_1(x) - f_2(x)$ cannot cross the $x-$axis from above, it can only cross the $x-$axis from below. We know that there must be at least one crossing, because f_1 and f_2 are continuous and both integrate to one. Hence, the function $f_1(x) - f_2(x)$ is non-positive until some point ang non-negative after some point until X_1. Morover, there are segments of strict positivity ang strict negativity. Hence,

$$\int_0^{X_1} (f_1(x) - f_2(x))x^2 dx > 0$$

This completes the proof. □

Proof. (of Corollary 2) Fix X and let $\Lambda_1(x) \equiv \lambda_1$ on $(0, X)$ correspond to the Calvo$^+$ model. The other hazard function, Λ_2, is at least somewhere strictly increasing. We claim it cannot be that $\Lambda_2(x) \geq \lambda_1$ for all x. Assume toward a contradiction that this is the case.

Then it cannot be that the graph of $f_2(x) - f_1(x)$ crosses the $x-$axis from below on $(0, X)$. If it does, there is a segment $[a, b]$ such that $f_2(x) - f_1(x)$ is positive to the right of b. But then the graph of $f_2(x) - f_1(x)$ never crosses the $x-$axis on $(b, X]$ again, because if it did cross it at some $d > b$, we would have $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (b, d), implying $f_2''(x) > f_1''(x)$ on (b, d), which contradicts $f_2'(b) \geq f_1'(b)$ and $f_2'(d) \leq f_1'(d)$ holding simultaneously. But we know that $f_1(X) = f_2(X) = 0$, which yields a contradiction.

Neither can it be that the graph of $f_2(x) - f_1(x)$ crosses the $x-$axis from above on $(0, X)$. If it does, there is a segment $[a, b]$ such that $f_2(x) - f_1(x)$ is positive to the left of a. But then the graph of $f_2(x) - f_1(x)$ never crosses the $x-$axis on $[0, a)$ again, because if it did cross it at some $c < a$, we would have $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (c, a), implying $f_2''(x) > f_1''(x)$ on (c, a), which contradicts $f_2'(a) \leq f_1'(a)$ and $f_2'(c) \geq f_1'(c)$ holding simultaneously. Hence, $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (c, a), implying $f_2''(x) > f_1''(x)$ on (c, a). But together with $f_2'(a) \leq f_1'(a)$ this contradicts $f_1'(0) = f_2'(0)$.

Hence, the graph of $f_2(x) - f_1(x)$ does not cross the $x-$axis from above or below on $(0, X)$. But Λ_2 is not identically equal to λ_1, so f_2 cannot coincide with f_1 everywhere. This yields the contradiction. Now we know that $\Lambda_2(x) < \lambda_1$ for some x. Since Λ_2 is non-decreasing, the conditions of Proposition 8 are satisfy, and Λ_1 generates a higher kurtosis of price changes. This completes the proof. □

Proof. (of Corollary 3) Let $X_1 > X_2$ and let Λ_1 and Λ_2 be constants λ_1 and λ_2 on their intervals. We claim that $\lambda_1 > \lambda_2$. Assume toward the contradiction $\lambda_1 \leq \lambda_2$. We know that the graph of the function $f_1(x) - f_2(x)$ must cross the $x-$axis from below at some point, because $f_1(X_2) > 0$,

$f_2(X_2) = 0$, and both f_1 and f_2 integrate to one. Hence, there is a point a such that $f_1(x) < f_2(x)$ to the left of a. Then the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis on $(0, a)$ again, since if it did there would be a point $c < a$ such that on (c, a) we have $f_1(x) < f_2(x)$ and hence $\Lambda 1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$ everywhere on (c, a). The latter contradicts $f_1'(a) \geq f_2'(a)$ and $f_1'(c) \leq f_2'(c)$ holding simultaneously.

But that the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis on $(0, a)$ again means that $f_1(x) < f_2(x)$ and hence $\Lambda 1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$ everywhere on $(0, a)$. Together with $f_1'(a) \geq f_2'(a)$ this contradicts $f_1'(0) = f_2'(0)$. Hence, $\lambda_1 > \lambda_2$. The pair Λ_1 and Λ_2 thus qualify for the Proposition 8, and Λ_1 generates a higher kurtosis of price changes. Hence, within the space of constant hazard functions with barriers higher X generate higher Kurtoses. By Proposition 3, the kurtosis for $X = \infty$ is the limit of any sequence generated by constant hazard functions with $X_k \to \infty$. Without loss of generality, the sequence can be constructed as monotone, so the kurtosis for $X = \infty$ is higher then any its element. But the kurtosis for an arbitrary Λ is majorized by that corresponding to a constant $\tilde{\Lambda}$ with the same barrier. Hence, the kurtosis for a constant Λ and $X = \infty$ is the highest possible one. This completes the proof. $\square$

Proof. (of Corollary 4) If the two hazard functions have the same curvature $k(x)$, it means that

$$\Lambda_1(x) = \Lambda_1(0) + \Lambda_1'(0) \int_0^x e^{\int_0^z \frac{k(w)}{w} dw} dz$$

$$\Lambda_2(x) = \Lambda_2(0) + \Lambda_2'(0) \int_0^x e^{\int_0^z \frac{k(w)}{w} dw} dz$$

We have $\mathcal{C}_1 > \mathcal{C}_2$ if and only if $\Lambda_1(0) > \Lambda_2(0)$. Using the same method as in the proof of Corollary 2, we can show that, since the frequency of adjustment is the same, there exists a $z < X$ such that $\Lambda_1(z) < \Lambda_2(z)$. Hence, $\Lambda_1'(0) < \Lambda_2'(0)$, and $\Lambda_1(x) - \Lambda_2(x)$ is a decreasing function. The graphs of $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ thus only cross once, so they qualify for Proposition 8, and $Kurt_1(\Delta p) > Kurt_2(\Delta p)$. $\square$

Proof. (of Proposition 9) Fix $\nu \geq 0$. In Lemma 3, we know that s increases in ρ, so it is sufficient to show that $Kurt(\Delta p)$ also does. For this purpose, take some $\rho_1 = 2\kappa_1 X_1^2/\sigma_1^2$. They generate $f_1(\cdot)$ with

$$\kappa_1 \left(\frac{x}{X_1}\right)^\nu f_1(x) = \frac{\sigma_1^2}{2} f_1''(x)$$

Now we want to increase ρ_1 to some $\rho_2 > \rho_1$. This can induce multiple $f_2(\cdot)$, since the distribution of price gaps also depends on X and σ^2. But the kurtosis of price changes only depends on ρ, so it suffices to show that one of the densities $f_2(\cdot)$ corresponding to ρ_2 generates a higher $Kurt(\Delta p)$. Let the new ρ_2 and the density $f_2(\cdot)$ be such that $\sigma_1^2 = \sigma_2^2$ and $f_1'(0) = f_2'(0)$. To compare the Kurtosis in this case it is enough to evaluate the sign of

$$\int_0^{X_2} f_2(x)x^2 dx - \int_0^{X_1} f_1(x)x^2 dx = \int_0^{X_2} (f_2(x) - f_1(x))x^2 dx$$

First, from the proof of Lemma 3 we know that $\hat{p}_2'(0) < \hat{p}_1'(0)$, which implies $X_2 > X_1$ because $f_2'(0) = f_1'(0)$. This, in turn, implies that $f_2(x) - f_1(x)$ is positive on (a, X_2) for some $a < X_1$.

Since $f_1(\cdot)$ and $f_2(\cdot)$ integrate to the same number over their supports, there must be a crossing b, to the left of which $f_1(x) > f_2(x)$. At this crossing, $f_1'(b) \geq f_2'(b)$. Now we will argue that there is no other crossing $c < b$.

Suppose, by way of contradiction, such a crossing exists. We have $f_2'(c) \leq f_1'(c)$ Subtract one Kolmogorov forward equations from the other:

$$x^\nu \left[\frac{\kappa_2}{X_2^\nu} f_2(x) - \frac{\kappa_1}{X_1^\nu} f_1(x)\right] = \frac{\sigma^2}{2}[f_2(x) - f_1(x)]'' \tag{62}$$

Now there are two options: $\kappa_2/X_2^\nu \geq \kappa_1/X_1^\nu$ or $\kappa_2/X_2^\nu < \kappa_1/X_1^\nu$. In the first case, since $f_2'(c) \leq f_1'(c)$ and $f_2(x) > f_1(x)$ to the left of c, from equation (62) we can conclude that $f_2''(x) > f_1''(x)$ for $x < c$, and hence $f_2'(x) - f_1'(x)$ only increases as x decreases. But this contradicts $f_1'(0) = f_2'(0)$. In the second case, since $f_2'(c) \leq f_1'(c)$ and $f_2(x) < f_1(x)$ to the right of c, from equation (62) we can conclude that $f_2''(x) < f_1''(x)$ for $x > c$, and hence $f_2'(x) - f_1'(x)$ only decreases as x decreases. But this contradicts $f_2'(b) > f_1'(b)$. Hence, there is no crossing to the left of b.

This means that $f_2(x) - f_1(x)$ is negative on $[0, b)$ and positive on (b, X_2). Since it integrates to zero over this whole interval, its integral with any positive increasing function (such as x^2) is positive. Hence, the kurtosis is higher for $\rho_2 > \rho_1$. $\square$

LEMMA 3. Consider two triplets $\{\sigma, X, \Lambda\}$ such that both generate the function $\hat{\Lambda}(\cdot)$ and the parameter ρ by using equation (85). The two triplets have the same Kurtosis of price changes $Kurt(\Delta p)$ and the same share of adjustment in the interior s. Furthermore,

$$\begin{aligned} N_a &= \frac{\sigma^2}{X^2}\hat{n}(\rho) \\ \frac{Kurt(\Delta p)}{6N_a} &= \frac{X^2}{\sigma^2}\frac{\hat{m}(\rho)}{6} \\ s &= \hat{s}(\rho) \end{aligned}$$

where $\hat{n}(\rho)$, $\hat{m}(\rho)$ and $\hat{s}(\rho)$ only depend on $\hat{\Lambda}(\cdot)$ and ρ. Moreover, $\hat{n}(\cdot)$ is increasing in ρ, $\hat{m}(\cdot)$ is decreasing in ρ, $\hat{s}(\cdot)$ is increasing in ρ, and $\hat{n}(0) = \hat{m}(0) = \hat{s}(0) = 1$.

Proof. (of Lemma 3) By the definition of $\hat{f}(\cdot)$, we have

$$\begin{aligned} \hat{f}(z) &= Xf(zX) \\ \hat{f}'(z) &= X^2 f'(zX) \end{aligned}$$

The function $\hat{f}(\cdot)$ itself is derived from

$$\rho\hat{\Lambda}(z)\hat{f}(z) = \hat{f}''(z) \text{ with } \hat{f}(1) = 0 \text{ and } \int_0^1 \hat{f}(z)dz = \frac{1}{2}$$

Computing the Kurtosis,

$$Kurt(\Delta p) = \frac{12N_a}{\sigma^2}\int_0^X f(x)x^2dx = -12f'(0)\int_0^X f(x)x^2dx = -12\hat{f}'(0)\int_0^1 \hat{f}(z)z^2dz$$

Since $\hat{f}(\cdot)$ is completely determined by ρ and $\hat{\Lambda}(\cdot)$, this quantity does not depend on other parameters. The share of adjustment between the boundaries is

$$s = 1 - \frac{f'(X)}{f'(0)} = 1 - \frac{\hat{f}'(1)}{\hat{f}'(0)} \tag{63}$$

It also only depends on $\hat{\Lambda}(\cdot)$ and ρ. The frequency of price changes is given by

$$N_a = -\sigma^2 f'(0) = -\frac{\sigma^2}{X^2}\hat{f}'(0)$$

From this we have $\hat{n}(\rho) = -\hat{f}'(0)$, so $\hat{n}(\rho)$ only depends on $\hat{\Lambda}$ and ρ. In the case when $\rho = 0$ the Kolmogorov forward equation is solved by a linear $\hat{f}(\cdot)$, and the slope is -1 from the boundary condition and the normalization. Hence, $\hat{n}(0) = 1$. Now take the other statistic:

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{2X^2}{\sigma^2}\int_0^1 \hat{f}(z)z^2dz = \frac{X^2}{6\sigma^2}\hat{m}(\rho)$$

Here the function $\hat{m}(\rho)$ is twelve times the integral of $\hat{f}(z)z^2$ which only depends on $\hat{\Lambda}(\cdot)$ and ρ. In the case when $\rho = 0$ we have $\hat{f}(z) = 1 - z$ for $z \in [0, 1]$ and hence $\hat{m}(0) = 1$.

Now fix the shape $\hat{\Lambda}(\cdot)$. Consider two different values of ρ: $\rho_1 > \rho_2$. They generate two distributions $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$. Taking the difference between the Kolmogorov forward equations that define them,

$$\hat{\Lambda}(z)(\rho_1\hat{f}_1(z) - \rho_2\hat{f}_2(z)) = (\hat{f}_1(z) - \hat{f}_2(z))''$$

It holds that $\hat{f}_1(1) = \hat{f}_2(1)$, so there must be another point $y \in (0, 1)$ where $\hat{f}_1(y) = \hat{f}_2(y)$, because $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$ integrate to the same number. Moreover, this point must be a crossing, meaning that $\hat{f}_1(z) - \hat{f}_2(z)$ has different signs on to the left and to the right of it. Suppose $\hat{f}_1(z) - \hat{f}_2(z)$ is positive to the right of y. This means $\hat{f}_1'(y) - \hat{f}_2'(y) \geq 0$. But then to the right of y it holds that $\hat{f}_1'(z) - \hat{f}_2'(z) > 0$, since the left-hand side of equation (63) is positive. Hence, the difference between $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$ only increases to the right of y, and they cannot cross again at $z = 1 > y$. This is a contradiction. The crossing is therefore such that $\hat{f}_1'(y) - \hat{f}_2'(y) \leq 0$. But then to the left of y it holds that $\hat{f}_1'(z) - \hat{f}_2'(z) < 0$, since the right-hand side of equation (63) is positive in this region. The difference between $\hat{f}_1(z)$ and $\hat{f}_2(z)$ increases as z decreases, as does he difference between $\hat{f}_1'(z)$ and $\hat{f}_2'(z)$. Hence, the crossing is unique and $\hat{f}_1'(0) < \hat{f}_2'(0)$. Moreover, $\hat{f}_1(z) - \hat{f}_2(z) > 0$ for $z \in [0, y)$ and $\hat{f}_1(z) - \hat{f}_2(z) < 0$ for $z \in (y, 1)$. From the latter fact together with $\hat{f}_1(1) = \hat{f}_2(1)$ it follows that $\hat{f}_1'(1) > \hat{f}_2'(1)$. To summarize:

- there is a unique $y \in (0, 1)$ such that $\hat{f}_1(z) - \hat{f}_2(z) > 0$ for $z \in [0, y)$ and $\hat{f}_1(z) - \hat{f}_2(z) < 0$ for $z \in (y, 1)$;
- $\hat{f}_1'(0) < \hat{f}_2'(0)$
- $\hat{f}_1'(1) > \hat{f}_2'(1)$

From the first bulletpoint it follows that $\hat{m}(\cdot)$ decreases in ρ. This is because $\hat{f}_1(\cdot) - \hat{f}_2(\cdot)$ integrates to zero over $(0, 1)$. Since it is positive until some z and negative afterwards, its integral with increasing positive functions (such as z^2) is always negative. From the second bulletpoint it follows that $\hat{n}(\cdot)$ increases in ρ, because $\hat{n}(\rho_i) = -\hat{f}_i'(0)$. From the second and the third bulletpoints combined it follows that s increases in ρ, because $\hat{f}'(1)$ and $\hat{f}'(0)$ are both negative, so their ratio decreases with ρ. This completes the proof. $\square$

Proof. (of Proposition 10) First, observe that if $v(x;\mu)$ and $\{\underline{x}(\mu), x^*(\mu), \overline{x}(\mu)\}$ represent a solution to the firm's problem with drift μ, then $\underline{x}(\mu) = -\overline{x}(-\mu)$, $x^*(\mu) = -x^*(-\mu)$, $\overline{x}(\mu) = -\underline{x}(-\mu)$, and $v(x;\mu) = v(-x;-\mu)$. This can be verified directly by plugging. Hence, $\Lambda(x;\mu) = \Lambda(-x;-\mu)$, because $\Lambda(x;\mu) = \kappa G(v(x;\mu) - v(x^*(\mu);\mu))$.

Second, observe that if $f(x;\mu)$ solves the Kolmogorov forward equation for μ and $\Lambda(x;\mu)$ then $f(x;\mu) = f(-x;-\mu)$. This can again be verified directly by plugging and using $\Lambda(x;\mu) = \Lambda(-x;-\mu)$. An implication of this symmetry is that $f'(x;\mu) = -f'(-x;-\mu)$. Hence, for the adjustment frequency we can write

$$
\begin{aligned}
N_a(\mu) &= \int_{\underline{x}(\mu)}^{\overline{x}(\mu)} \Lambda(x;\mu) f(x;\mu) dx + \frac{\sigma^2}{2} f'(\underline{x}(\mu);\mu) - \frac{\sigma^2}{2} f'(\overline{x}(\mu);\mu) && (64)\\
&= \int_{-\overline{x}(-\mu)}^{-\underline{x}(-\mu)} \Lambda(-x;-\mu) f(-x;-\mu) dx - \frac{\sigma^2}{2} f'(-\underline{x}(\mu);-\mu) + \frac{\sigma^2}{2} f'(-\overline{x}(\mu);-\mu) && (65)\\
&= \int_{\underline{x}(-\mu)}^{\overline{x}(-\mu)} \Lambda(x;-\mu) f(x;-\mu) dx + \frac{\sigma^2}{2} f'(\underline{x}(-\mu);-\mu) - \frac{\sigma^2}{2} f'(\overline{x}(-\mu);-\mu) = N_a(-\mu) && (66)
\end{aligned}
$$

In a similar vein, using $q(x) = \Lambda(x) f(x)/N_a$ and hence $q(x;\mu) = q(-x;-\mu)$, we can write for any even moment of $Q(\cdot)$

$$
\mathbb{E}\left[\Delta p^{2k}\right](\mu) = \int_{\underline{x}(\mu)}^{\overline{x}(\mu)} x^{2k} q(x;\mu) dx = \int_{-\overline{x}(-\mu)}^{-\underline{x}(-\mu)} x^{2k} q(-x;-\mu) dx = \int_{\underline{x}(-\mu)}^{\overline{x}(-\mu)} x^{2k} q(x;-\mu) dx = \mathbb{E}\left[\Delta p^{2k}\right](-\mu)
$$

This holds for the fourth moment and variance, so it holds for Kurtosis as well. Hence, both $Kurt(\Delta p)$ and N_a are symmetric in μ. They are also analytical functions of μ and can be written as

$$
\begin{aligned}
Kurt(\Delta p) &= \sum_{i=0}^{\infty} a_i \mu^i && (67)\\
N_a &= \sum_{i=0}^{\infty} b_i \mu^i && (68)
\end{aligned}
$$

The odd terms in these infinite sums must be zero, meaning

$$
\begin{aligned}
Kurt(\Delta p) &= a_0 + o(\mu^2) && (69)\\
N_a &= b_0 + o(\mu^2) && (70)
\end{aligned}
$$

This completes the proof. $\square$

Proof. (of Proposition 11) The proof is in the statement of the proposition.

Proof. (of Proposition 12) Differentiating Ω

$$\Omega'(\delta) = X\, f(-X+\delta)\; + \int_{-X}^{-X+\delta} f(x)\, dx$$

taking $\delta \to 0$, since the invariant distribution satisfies $f(-X) = 0$, we have $\Omega'(0) = 0$.

Now we seek to characterize $\lim_{t\downarrow 0} \omega_\delta(t;\delta)$. We will show that $\lim_{t\downarrow 0} \omega_\delta(t;0) = \infty$ if $X < \infty$.

For this case we replace the initial condition by $f(x+\delta)$ by $f(x)+f'(x)\delta$ where f is the density of the invariant distribution. We can ommitt the contribution from the term $f(x)$, since it is equal to zero by virtue of being the invariant distribution.

The KFE gives the following properties:

1. For all $t > 0$, since $-X$ is an exit point, $f(-X,t) = 0$.
2. For all $t > 0$, there exists $\underline{x}(t) > -X$, so that $f(x,t) < f(x,0) = f'(x)\delta > 0$ for all $x \in [-X, \underline{x}(t)]$. This follows because $f(x,t)$ is differentiable in x and $f(-X,t) = 0$.
3. For all $x \in (-X, 0)$ we have: $f(x,t) \to f(x,0)$ as $t \downarrow 0$. This follows since $f(x,t)$ is differentiable in time t for all x.

From these properties we obtain that $f'(-X,t) \to \infty$ as $t \downarrow 0$. Hence, $\omega_\delta(0,0) = \infty$. $\square$

Proof. (of Proposition 13) The frequency of adjustment is given by

$$\begin{aligned} N_a &= \int_{-\infty}^{\infty} f(x)(\Lambda(0) + \kappa x^\nu) dx = \int_{-\infty}^{\infty} \tilde{f}(z) \left(\Lambda(0) + \kappa \left(\frac{z}{\eta}\right)^\nu\right) dz \\ &= \frac{\kappa}{\eta^\nu} \int_{-\infty}^{\infty} p(z)(\alpha + z^\nu) dz = \frac{\kappa}{\eta^\nu} \tilde{N}(\nu,\alpha) = \frac{\beta^2\eta^2}{2} \tilde{N}(\nu,\alpha) \end{aligned}$$

The flexibility index is

$$\begin{aligned} \mathcal{F} &= -\int_{-\infty}^{\infty} x(\Lambda(0) + \kappa x^\nu) f'(x) dx = -\int_{-\infty}^{\infty} z\left(\Lambda(0) + \kappa\left(\frac{z}{\eta}\right)^\nu\right) p'(z) dz \\ &= -\frac{\kappa}{\eta^\nu} \int_{-\infty}^{\infty} z(\alpha + z^\nu)\tilde{f}'(z) dz = \frac{\kappa}{\eta^\nu}\left(\int_{-\infty}^{\infty} \tilde{f}(z)(\alpha + z^\nu) dz + \nu \int_{-\infty}^{\infty} p(z) z^\nu dz\right) \\ &= \frac{\kappa}{\eta^\nu}\left(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha\right) = \frac{\beta^2\eta^2}{2}\left(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha\right) \end{aligned}$$

The distribution of price changes is given by

$$q(x) = \frac{f(x)(\Lambda(0) + \kappa x^\nu)}{N_a} = \frac{\eta\tilde{(}\eta x)(\alpha + (\eta x)^\nu)}{\tilde{N}(\nu,\alpha)}$$

To compute the kurtosis, we need the fourth moment and the variance:

$$\mathbb{E}[\Delta p^4] = \int_{-\infty}^{\infty} x^4 q(x)dx = \frac{1}{\eta^4 N(\nu,\alpha)} \int_{-\infty}^{\infty} z^4 p(z)(\alpha + z^\nu)dz$$
$$\mathbb{E}[\Delta p^2] = \int_{-\infty}^{\infty} x^2 q(x)dx = \frac{1}{\eta^2 N(\nu,\alpha)} \int_{-\infty}^{\infty} z^2 p(z)(\alpha + z^\nu)dz$$

These expressions imply that $\mathbb{E}[\Delta p^4]/\mathbb{E}[\Delta p^2]^2$ only depends on (ν, α). $\square$

Proof. (of Proposition 14) Let $f_1(x)$ and $f_2(x)$ be the price gap distributions generated by $\Lambda_1(x)$ and $\Lambda_2(x)$. Assume without loss that $\kappa_1 < \kappa_2$. We will first prove that $\Lambda_1(0) > \Lambda_2(0)$ whenever N_a is the same in the two models. That $Kurt_1(\Delta p) > Kurt_2(\Delta p)$ will then follow from Proposition 8. Finally, we will show that $\mathcal{F}_1 < \mathcal{F}_2$.

(1) Suppose by contradiction that $\Lambda_1(0) \leq \Lambda_2(0)$. Then, $\Lambda_1(x) < \Lambda_2(x)$ for all $x > 0$. Since N_a and σ^2 are the same in the two models, we know that $f_1'(0) = f_2'(0)$.

Suppose there is a point $a > 0$ at which the graph of $f_1(x)$ crosses that of $f_2(x)$ from below. That is, $f_1(a) = f_2(a)$ and $f_1(x) < f_2(x)$ to the left of a. Then the graphs of $f_1(x)$ and $f_2(x)$ never cross again to the left of a. If they did cross at some point $b < a$, we would have $f_1'(a) \geq f_2'(a)$ and $f_1'(b) \leq f_2'(b)$, so that $f_1'(a) - f_1'(b) \geq f_2'(a) - f_2'(b)$, but this is impossible, since $f_1(x) < f_2(x)$ and $\Lambda_1(x) < \Lambda_2(x)$ on (a, b), while $\sigma^2 f_i''(x)/2 = \Lambda_i(x) f_i(x)$ for $i \in \{1, 2\}$. Hence, $f_1(x) < f_2(x)$ for all $x < a$, which contradicts $f_1'(0) = f_2'(0)$ for the same reason.

Suppose there is a point $c > 0$ at which the graph of $f_1(x)$ crosses that of $f_2(x)$ from above. That is, $f_1(c) = f_2(c)$ and $f_1(x) < f_2(x)$ to the right of c. Then the graphs of $f_1(x)$ and $f_2(x)$ never cross again to the right of c. If they did cross at some point $d > c$, we would have $f_1'(d) \geq f_2'(d)$ and $f_1'(c) \leq f_2'(c)$, so that $f_1'(d) - f_1'(c) \geq f_2'(d) - f_2'(c)$, but this is impossible, since $f_1(x) < f_2(x)$ and $\Lambda_1(x) < \Lambda_2(x)$ on (c, d), while $\sigma^2 f_i''(x)/2 = \Lambda_i(x) f_i(x)$ for $i \in \{1, 2\}$. Hence, $f_1(x) < f_2(x)$ for all $x > c$, which contradicts $f_1'(x) - f_2'(x) \longrightarrow 0$ as $x \longrightarrow \infty$ for the same reason.

By what was said above, the graphs of $f_1(x)$ and $f_2(x)$ cannot cross, but they must, since these functions integrate to the same number and have the same limit at infinity. Hence, $\Lambda_1(0) \leq \Lambda_2(0)$ is impossible when σ^2 and N_a are the same in the two models.

(2) Now since $\kappa_1 < \kappa_2$ and $\Lambda_1(0) > \Lambda_2(0)$, the two generalized hazard functions $\Lambda_1(x)$ and $\Lambda_2(x)$ satisfy the conditions of Proposition 8. From this it follows that $Kurt_1(\Delta p) > Kurt_2(\Delta p)$.

(3) The flexibility index for the power-plus case is given by

$$\mathcal{F} = \int_{-\infty}^{\infty} f(x)(\Lambda(x) + \Lambda'(x)x)dx = (1 + \nu)N_a - \nu\Lambda(0)$$

Since the two models deliver the same N_a and ν is fixed, the one with a greater intercept has a smaller $\mathcal{F}$. This completes the proof. $\square$

Proof. (of Proposition 15). We will make two observations, one about Λ and one about F_n required to establish the two main results of the proposition. Then we will use Lemma 4 finish the proof.

The first observation is that the symmetry of Λ around $x = 0$ implies that all the odd numbered derivatives evaluated at $x = 0$ of Λ are equal to zero.

The second observation is a property of the function $F_n(x)$ generated by the recursion in equation (60), which can be written as:

$$F_n(x) = \tilde{F}_n(x) - \left(\frac{\sigma^2}{2}\right)^{n-1} \frac{\partial^{2n-2}\Lambda(x)}{\partial x^{2n-2}}$$

where $\tilde{F}_n(\cdot)$ depends only on the level of $\Lambda(\cdot)$ and at most the first $2n-1$ derivatives of $\Lambda(\cdot)$, evaluated at x. This property can be established by induction. It is true for $F_1(x) = -\Lambda(x)$ for $n = 1$. Now assume it holds for n, and we will show that it holds $n+1$. To do so we compute F_{n+1} according to the recursion. On this computation, the first term is the product of $\sigma^2/2$ times the sum of the second derivative of $\tilde{F}_n(x)$ with respect to x and of the second derivative of $-\left(\sigma^2/2\right)^{n-1} \partial^{2n-2}\Lambda(x)/\partial x^{2n-2}$ with respect to x. The remaining term, $-\Lambda(x)F_n(x)$, involves no derivatives. This finishes the induction step, and thus established the desired result for F_n.

1. If we know the function $\Lambda(x)$, then we can recursively compute $F_n(x)$ from equation (60). Evaluating this expressions at $x = 0$ and using equation (61) we obtain all the derivatives of S evaluated at $t = 0$. In particular, these expressions only use the level and the even derivatives of Λ evaluated at $x = 0$.If S is analytical, the expansion of S at $t = 0$ gives the values everywhere.

2. If we know the function S, we can take all its derivatives at $t = 0$, and by equation (61) we know all the values of $F_n(0)$ for $n \geq 1$. Next we argue that the recursion in equation (60) evaluated at $x = 0$, will give us all the even order derivatives of Λ evaluated at $x = 0$. Since Λ is symmetric, all the derivatives of odd order, evaluated at $x = 0$, so we are only interested in the even derivatives at $x = 0$. Next we argue that, algorithmically, we can recursively recover the derivatives up to order $2n-2$ with $\{F_n(0)\} for j = 1, \ldots, 2n-2$. First we note that $\Lambda(0)$ and $\Lambda''(0)$ are given by $F_1(0)$ and $F_2(0)$. Now assume we know all the derivatives up to order $2n-2$. Then, given the value of $\partial^{n+1}S(0)/\partial t^{n+1} = F_{n+1}(0)$, the known values for $\Lambda(0)$, $F_n(0)$, and $\sigma^2/$, using the recursion we obtain the implied value for $\partial_{xx}F_n(0)$. Using that F_n depend at most on $2n-2$ derivatives of Λ, as well as the particular expression derived above, we obtain the value of $\partial^{2n}\Lambda(0)/\partial t^{2n}$. This completes the induction step, and hence establishes the desired property, and hence the level and all the derivatives of Λ at $x = -0$ have been recovered. Finally, since Λ is assumed to be analytical, an expansion around $x = 0$ gives its value at any other x.

This completes the proof. □

C Estimation and measurement issues

In this appendix we present our estimation algorithm and some additional results. The next proposition shows that if we have a sample with mixed N different type of products all with the same kurtosis but with different variance, then the kurtosis of the price changes of such a mixture is higher than the kurtosis for each of them.

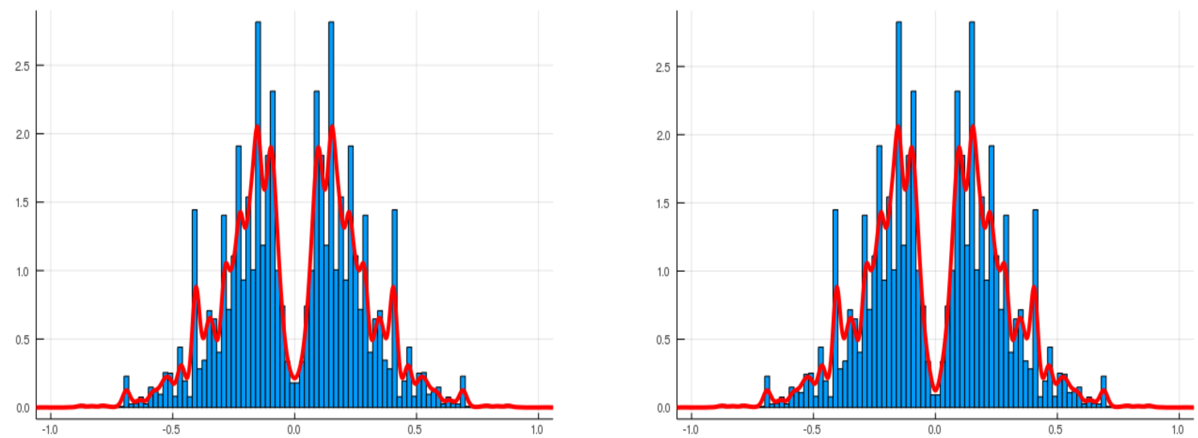
PROPOSITION 11. Assume that Δp is a mixture of N distributions, with weights $\{\omega_j\}_{j=1}^N$. Assume that for each distribution j, price changes have the same kurtosis K, but they may have different variance V_j. Then

$$Kurt(\Delta p) = \frac{\sum_j \omega_j K V_j^2}{\left[\sum_j \omega_j V_j\right]^2} = K \frac{\sum_j \omega_j V_j^2}{\left[\sum_j \omega_j V_j\right]^2} = K \frac{\sum_j J(V_j)\omega_j}{J\left(\sum_j V_j \omega_j\right)} \geq K$$

with strict inequality if the distribution of $\{V_j\}_{j=1}^N$ is not degenerate, since $J(V) = V^2$ is a strictly convex function.

The proof is contained in the statement of the proposition. Next, we plot the symmetrized histograms with fitted densities for two data cleaning procedures: the one that eliminates price changes smaller then 2 cents in absolute value, and the one eliminating those smaller than 1 cent in absolute value. The distributions are very close, with immaterial differences in the bars around zero.

Figure 6: Distribution of price changes in a narrow category



Pooling all products for category 561 "Non-durable household goods"

We use the method of moments to estimate the mixture of two Gamma distributions with the parameters ω (the weight), (α_1, β_1) and (α_2, β_2). The moments of $|\Delta \tilde{p}_t|$ we use are denoted by $\gamma_{j,k}$:

$$\gamma_{j,k} = \frac{\mathbb{E}[|\Delta \tilde{p}_t|^{j+k}]}{\mathbb{E}[|\Delta \tilde{p}_t|^j]\mathbb{E}[|\Delta \tilde{p}_t|^k}$$

For a mixture of two Gamma distributions with the weight ξ on the first one,

$$\gamma_{j,k} = \frac{\left[\beta_2^{j+k}\omega \frac{\Gamma(\alpha_1 + j + k)}{\Gamma(\alpha_1)} + \beta_1^{j+k}(1-\omega)\frac{\Gamma(\alpha_2 + j + k)}{\Gamma(\alpha_2)}\right]}{\left[\beta_2^j \omega \frac{\Gamma(\alpha_1 + j)}{\Gamma(\alpha_1)} + \beta_1^j(1-\omega)\frac{\Gamma(\alpha_2 + j)}{\Gamma(\alpha_2)}\right]\left[\beta_2^k \omega \frac{\Gamma(\alpha_1 + k)}{\Gamma(\alpha_1)} + \beta_1^k(1-\omega)\frac{\Gamma(\alpha_2 + k)}{\Gamma(\alpha_2)}\right]} \tag{71}$$

Using these moments allows us to recover ω, α_1, α_2, and the ratio β_1/β_2. The exact values of β_1 and

β_2 are pinned down by the normalization $\mathbb{E}[|\Delta\tilde{p}_t|] = 1$. To estimate $\gamma_{j,k}$, we rely on Proposition 5:

$$\frac{\mathbb{E}[|\Delta\tilde{p}_t|^{j+k}]}{\mathbb{E}[|\Delta\tilde{p}_t|^{j}]\mathbb{E}[|\Delta\tilde{p}_t|^{k}} = \frac{\mathbb{E}[|\Delta p_{it}|^{j+k}]}{\mathbb{E}[|\Delta p_{it}|^{j}|\Delta p_{is}|^{k}]}$$

For all seven product categories, we get four moments ($\hat{\gamma}_{1,1}$, $\hat{\gamma}_{2,1}$, $\hat{\gamma}_{3,1}$, and $\hat{\gamma}_{3,2}$) from the data and solve the system of four analogs of equation (71). We minimize the sum of deivations squared with equal weights. The results are presented in Table 2.

Category	$\hat{\gamma}_{11}$	$\hat{\gamma}_{21}$	$\hat{\gamma}_{31}$	$\hat{\gamma}_{32}$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1/\hat{\beta}_2$	$\hat{\omega}$	$\hat{\alpha}_{22}$
111	1.248	1.406	1.507	1.787	2.099	12.190	228.677	0.161	4.248
119	1.282	1.507	1.702	2.381	1.058	6.012	91.439	0.109	3.747
1212	1.242	1.476	1.786	2.9230	0.599	3.873	73.414	0.000	4.151
122	1.243	1.397	1.508	1.903	1.848	9.779	173.048	0.131	4.460
118	1.289	1.539	1.777	2.552	3.123	9.836	0.628	0.580	3.610
117	1.281	1.511	1.721	2.484	0.967	5.442	84.154	0.089	3.801
561	1.216	1.394	1.586	2.271	0.998	5.783	103.470	0.031	4.782

Table 2: Moments taken from the data and the estimated parameters

Specializing to the case with a single Gamma distribution $\omega = 1$ allows us to recover the expressions for α in closed form. Consider $\gamma_{j,1}$ for some j:

$$\gamma_{j,1} = \frac{\Gamma(\alpha + j + 1)\Gamma(\alpha)}{\Gamma(\alpha + j)\Gamma(\alpha + 1)} = 1 + \frac{j}{\alpha}$$

Hence,

$$\alpha = \frac{j}{\gamma_{j,1} - 1} \tag{72}$$

Since we attach particula importance to the kurtosis, we would also like to use $\gamma_{2,2}$:

$$\gamma_{j,2} = \frac{\Gamma(\alpha + j + 2)\Gamma(\alpha)}{\Gamma(\alpha + j)\Gamma(\alpha + 2)} = \frac{(\alpha + j + 1)(\alpha + j)}{(\alpha + 1)\alpha} = \left(1 + \frac{j + 1}{\alpha}\right)\frac{\gamma_{j,1}}{\gamma_{1,1}}$$

This leads to

$$\alpha = \frac{(j + 1)\gamma_{j,2}}{\gamma_{j,2}\gamma_{1,1} - \gamma_{j,1}} \tag{73}$$

Notice that β, the scale of the distribution, drops out from these expressions, because $\gamma_{j,k}$ are dimensionless moments. We use a linear combinations of expressions in equation (72) and equation (73) with $\hat{\gamma}_{j,1}$ for $j \in \{1, 2\}$ and $\hat{\gamma}_{22}$ as estimators of α. Consistency requires the weights of the combinations to sum to one, and we make them inversely proportional to the bootstrapped variance of the estimators of summands. The estimates are presented in the last column of Table 2: the estimate $\hat{\alpha}_{22}$ is constructed from $\hat{\gamma}_{11}$, $\hat{\gamma}_{21}$, and $\hat{\gamma}_{22}$.

In Table 3 we present some additional statistics. First, we tabulate skewness of the distribution of price changes to show that the distributions are close to symmetric. Then, we contrast the estimates of the Kurtosis with the full sample and with the first two price changes only. The difference between them is suggestive of a strong correlation between consecutive price changes (squared), and of a weaker correlation between distant price changes. As can be seen from equation (27), how much the underlying Kurtosis is different from that of the pooled distribution (without accounting for product heterogeneity) increases with this correlation. The implied correlation and the coefficient of variation (present in equation (27) as well) are tabulated in the remaining two columns.

Category	Skewness	Kurtosis	Kurtosis ($t = 1, 2$)	Implied Correlation	$CV(\Delta\tilde{p}_t)$
111	-0.121	1.656	1.426	0.440	1.555
		(0.065)	(0.071)		
119	0.011	1.955	1.288	0.339	1.683
		(0.050)	(0.042)		
1212	-0.020	2.051	1.710	0.284	1.589
		(0.162)	(0.186)		
122	-0.025	1.677	1.189	0.390	1.398
		(0.051)	(0.019)		
118	-0.012	2.044	1.663	0.295	1.620
		(0.118)	(0.150)		
117	-0.004	1.989	1.422	0.303	1.577
		(0.047)	(0.089)		
561	-0.006	1.778	1.403	0.374	1.524
		(0.133)	(0.066)		

Table 3: Additional statistics

Now we present the estimation procedure to recover the flow cost function from Section 2.2. The model in this section permits Λ to be unbounded. We take advatage of that and work with a power hazard $\Lambda(x) = \kappa x^{\nu}$. This form of Λ gives rise to a specifica functional form of Q. We compute the moments of Q as functions of (κ, ν) and then estimate them using the mothod of moments.

Suppose $\Lambda(x) = \kappa x^{\nu}$. Denote $\rho = 2\kappa/\sigma^2$. The corresponding density of price gaps has to obey a Kolmogorov forward equation that has the form

$$\rho x^{\nu} f(x) = f''(x)$$

With $X = \infty$, the solution is

$$f(x) = \frac{x^{1/2}\mathcal{K}_{1/(\nu+2)}\left(\frac{2\sqrt{\rho}}{\nu+2}x^{(\nu+2)/2}\right)}{2\int\limits_0^{\infty} x^{1/2}\mathcal{K}_{1/(\nu+2)}\left(\frac{2\sqrt{\rho}}{\nu+2}x^{(\nu+2)/2}\right)dx}$$

The distribution of price changes is then given by

$$q(-x) = \frac{\kappa x^{\nu} f(x)}{N_a} = \frac{\kappa x^{\nu+1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right)}{2N_a \int\limits_0^{\infty} x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx}$$

Since $\mathbb{V}[\Delta\tilde{p}_t] = 1$, we have $\sigma^2 = N_a$, so

$$q(-x) = \frac{\kappa x^{\nu} f(x)}{N_a} = \frac{\rho x^{\nu+1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right)}{4 \int\limits_0^{\infty} x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx}$$

This has to be a probability distribution, so it integrates two one. We also have the moment condition $\mathbb{E}[(\Delta\tilde{p}_t)^4] = Kurt(\Delta\tilde{p}_t)$. Writing the two restrictions in a convenient form,

$$\int\limits_0^{\infty} (\rho x^{\nu} - 2) x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx = 0$$

$$\int\limits_0^{\infty} (\rho x^{\nu+4} - 2Kurt(\Delta\tilde{p}_t)) x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx = 0$$

From these two relations we can get $(\hat{\rho}, \hat{\nu})$. After that, using $\sigma^2 = N_a$, we can recover $\hat{\kappa}$:

$$\hat{\kappa} = \frac{\hat{\rho} N_a}{2}$$

The system of two restrictions can be solved exactly, and the model is just identified. The results for the category 561 (”non-durable household goods”) are presented on Figure 7. The estimated parameters are $\hat{\nu} = 2.285$ and $\hat{\kappa} = 30.747$, corresponding to the Kurtosis 1.64, slightly below the quadratic case.

D Flexibility Index: scope and limitations

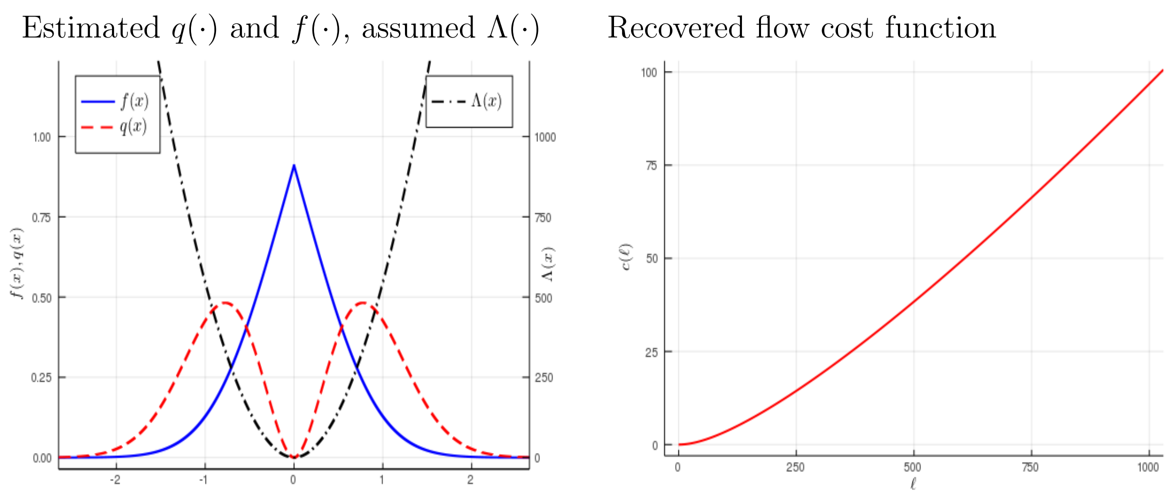
The impulse response function (IRF) of the aggregate price level after a shock δ can be written as

$$\mathcal{P}(t, \delta) = \Omega(\delta) + \int_0^t \omega(s, \delta)\, ds$$

where $\omega(s, \delta)$ is the flow contribution to the IRF at time $s > 0$, and $\Omega(\delta)$ is the time $t = 0$ jump in the price level. By definition $\frac{\partial}{\partial t}\mathcal{P}(t, \delta) = \omega(t, \delta)$. The flow value of the IRF of the aggregate price level at time $t > 0$ is given by

$$\omega(t, \delta) = -\int_{-X}^{X} x\Lambda(x) f(x, t) dx + X\sigma^2 \left[f'(-X, t) - f'(X, t) \right]$$

Figure 7: Estimated distribution of price changes and implied cost function



where $f(x,t)$ is the distribution of the price gaps among the firms that have not adjusted prices t units of time after the monetary shock. The first term is the change of prices across the distribution of price gaps at time t, with $f(x,t)$ solving the time dependent Kolmogorov Forward Equation:

$$\partial_t f(x,t) = -\Lambda(x) f(x,t) + \frac{\sigma^2}{2} \partial_{xx} f(x,t) \text{ for all } x \in [-X, X] \text{ and } t \geq 0, \tag{74}$$

$$f(X,t) = f(-X,t) = 0 \text{ for all } t > 0, \text{ and } f(x,0) = f_0(x) \text{ for all } x \in [-X, X] \tag{75}$$

The initial jump is given by

$$\Omega(\delta) = \int_{-X}^{-X+\delta} (-x+\delta) \ f_0(x) \, dx$$

The initial distribution f_0 that we consider is a uniform shift by δ of some distribution $\hat{f}$:

Assumption 2. The initial condition is $f_0(x) = \hat{f}(x+\delta)$, where $\hat{f}$ i) equals zero at the bounds, $0 = \hat{f}(-\bar{X}) = \hat{f}(\bar{X})$, ii) increases close to the lower bound, $0 < \hat{f}'(-\bar{X}) < \infty$, and iii) is differentiable on $(-\bar{X}, 0)$.

We write $f_0(x) = \hat{f}'(x)\delta + o(\delta)$ and consider the case of small δ. Note that the assumptions allow $\hat{f}$ to be the invariant distribution corresponding to $\{X, \Lambda, \sigma^2\}$, but they do not require it. In particular, $\hat{f}$ can be any distribution that has for any strictly positive time evolved according to equation (74) and equation (75). The Flexibility index is defined as $\mathcal{F} \equiv \frac{\partial}{\partial \delta} \omega(0,\delta)|_{\delta=0}$, which is equivalent to the definition in equation (17) in Caballero and Engel (2007).

Proposition 12. Let Ω and ω be the jump and flow values of the IRF of prices at $t = 0$. Let $X < \infty$, let Λ satisfy Assumption 1, and assume that the initial distribution f_0 satisfies Assumption 2. Then $\Omega(0) = \Omega'(\delta)|_{\delta=0} = 0$. Moreover, $\partial_\delta \omega(0,\delta)|_{\delta=0} = \infty$ and $\omega(0,0) = 0$. Thus, if $X < \infty$, the flexibility index is infinite for any Λ.

Because of this result we will move to analyze the flexibility index for models with $X = \infty$, where it is finite. We will will do so for a family of hazard functions which is a slight generalization of the one treated in Section 5.1.

D.1 Power plus family of generalized hazard functions

We consider a simple four parameter family of models where $\Lambda(x) = \Lambda(0) + \kappa x^{\nu}$. We label this case as power-plus, because it adds a constant to the power case. Besides $\Lambda(0)$, κ, and ν, the other parameter of the model is σ^2. We introduce the parameter η and let α be the adjusted intercept:

$$\eta = \left(\frac{2\kappa}{\sigma^2}\right)^{\frac{1}{\nu+2}} \quad , \quad \alpha = \frac{\Lambda(0)\eta^{\nu}}{\kappa} \quad .$$

The quadratic case is $\nu = 2$ and $\alpha = 0$. This adjusted intercept measures the relative magnitude of $\Lambda(0)$ and the slope κ, increasing in the former and decreasing in the latter. We will show that for a fixed power the Kurtosis, adjustment frequency, and the flexibility index only depend on α.

PROPOSITION 13. Fix σ^2 and let $\Lambda(x)$ be a power-plus hazard function parameterized by $(\kappa, \Lambda(0), \nu)$. The adjustment frequency, the kurtosis of price changes, and the flexibility index are

$$N_a = \frac{\eta^2\sigma^2}{2}\tilde{N}(\nu,\alpha) \quad , \quad \frac{Kurt(\Delta p)}{6N_a} = \frac{1}{\eta^2\sigma^2}\tilde{K}(\nu,\alpha) \quad , \quad \mathcal{F} = \frac{\eta^2\sigma^2}{2}(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha)$$

where $\tilde{N}(\nu,\alpha)$ and $\tilde{K}(\nu,\alpha)$ only depend on ν and α; $\tilde{N}(0,\alpha) \equiv 1+\alpha$, and $\tilde{K}(0,\alpha) \equiv 2/(1+\alpha)$.

With no intercept, the flexibility index and adjustment frequency are related by a simple formula via the elasticity of the hazard:

$$\mathcal{F} = N_a(1+\nu)$$

If two models have the same (ν, α), the density of price changes in one is a rescaling of that in the other. This implies that kurtosis (and other scale-free statistics) is the same. If η also coincides in the two models, the distributions of price changes are identical.

The power-plus parameterization allows us to illustrate substantial disconnect between the CIR and the flexibility index. In one example where we vary one parameter at time: in this case the flexibility index and the cumulative IRF move in the same direction. In the second example we change three parameters at a time and show how for the same flexibility index cumulative IRF can vary substantially, even keeping the adjustment frequency fixed.

PROPOSITION 14. Assume that Λ is given by a power-plus function. Fix (ν, σ^2) and take two different power plus generalized hazard functions Λ_1 and Λ_2. If they generate the same frequency N_a, then $Kurt_1(\Delta p) > Kurt_2(\Delta p)$ if and only if $\mathcal{F}_1 < \mathcal{F}_2$.

This result is not surprising, since we are varying one parameter only. This comparative static exercise is very far away from the idea of a "sufficient statistic", where one finds a statistic that summarizes significant outputs of a class of models. Even the simple power-plus parameterization affords much more flexibility than varying one parameter can offer.

Now we turn to the second case, where we argue that, however intuitive this might be, relying on the flexibility index can be quite misleading. In the right panel of Figure 5 we display several economies with the same adjustment frequency N_a, and with the same Flexibility Index $\mathcal{F}$, that feature very different cumulative response to a monetary shock. That is, we vary the parameters in such a way that both $\mathcal{F}$ and N_a stay constant, while $\mathcal{M}'(0)$ varies substantially. This is done by increasing the power parameter ν and finding the pairs $(\Lambda(0), \kappa)$ that keep N_a and $\mathcal{F}$ constant. We solve this problem numerically and find that for the same N_a and $\mathcal{F}$ the Kurtosis of price changes varies by 90% when ν increases from 2 to 20, as plotted in the figure. The slope of the impulse response at $t = 0$ does not capture the area under it in a reliable way.

In the left panel of Figure 5 we take two examples, one with $\nu = 2$ and the other with $\nu = 10$, and display the entire output impulse response function $Y(t)$ as a function of time t. Thus, both IRF's have the same frequency N_a and flexibility index $\mathcal{F}$. The areas under both IRF's are clearly different, the one for $\nu = 10$ is at least 50% larger than the one for $\nu = 2$, consistent with the values displayed in the right panel of the figure. By construction the slope of $Y(\cdot)$ at $t = 0$ is the same for both cases (i.e. for $\nu = 2$ and $\nu = 10$), since both IRF's have the same Flexibility index $\mathcal{F}$. Yet, the slopes of both impulse responses starts to differ substantially even for low values of t. Since in both cases $N_a = 1$, the values of time in the horizontal axis can be measured in terms of expected adjustment time. For instance, if prices change on average three times a year, meaning $N_a = 3$, then $t = 1$ represents 4 months. The ratio of the two IRF evaluated at $t = 1$ is higher than 4, namely $Y_{10}(1)/Y_2(1) \approx 4.4$. This example shows that even the short run output effect can be substantially different with the same flexibility index.

E Duration Analysis and Generalized Hazard Rate

In this section we consider the Survival and the Hazard Rate as functions of the *duration* of the price spells. Duration-based functions are often used in sticky price models. It is interesting to know whether the information encoded in them is different from that encoded in the size-distribution of price changes used above. We establish conditions for a non-trivial equivalence result: the distribution of durations and the variance of price changes together contain the same information about the fundamentals of the model as the distribution of price changes and frequency of adjustment. The distribution of spells with one statistic on the size of changes (the variance) is as informative as the size-distribution of changes and one temporal statistic (the frequency).

Denote by $S(t)$ the Survival function, the probability that a price spell lasts at least t units of time. We will show that, when $X = \infty$, an analytical Survival Function S uniquely identifies an analytical Generalized Hazard Rate function Λ. When $X = \infty$, the Survival function is given by

$$S(t) = \mathbb{E}\left[e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \text{ for all } t \geq 0 \tag{76}$$

where the expectation is taken with respect to the paths of the drift-less Brownian motion x with variance per unit of time equal to σ^2. The value of $S(t)$ is the Feynman-Kac formula evaluated at $x = 0$. The hazard rate $h(t) = -S'(t)/S(t)$ measures the probability per unit of time of a price spell ending conditional on lasting at least t. For example, the Survival function and its associated hazard rate for the case of a quadratic generalized hazard rate $\Lambda(x) = \Lambda(0) + \kappa x^2$ are:

$$S(t) = \frac{e^{-t\Lambda(0)}}{\left(\cosh\left(t\sqrt{2\kappa\sigma^2}\right)\right)^{\frac{1}{2}}} \text{ and } h(t) = \Lambda(0) + \sqrt{\kappa\frac{\sigma^2}{2}} \tanh\left(t\sqrt{2\kappa\sigma^2}\right) \text{ for all } t \geq 0 \tag{77}$$

This was obtained by Kac in his seminal study of what we now know as the Kac formula. The next lemma gives the main technical result to establish the link between the Survival function, which can in principle be measured in the data, and the generalized hazard function $\Lambda(x)$.

Lemma 4. Fix a value of $\sigma^2 > 0$, and assume that $X = \infty$. Assume that S is related to Λ by equation (76). The derivatives of the Survival function S a time $t = 0$ and the derivatives of Λ at $x = 0$ are related by the recursively generated functions $\{F_n\}$ as follows:

$$\frac{\partial^n S(t)}{\partial t^n}\Big|_{t=0} = F_n(0) \text{ and all } n = 1, 2, \dots \text{ where } F_n(\cdot) \text{ are given by}$$
$$F_{n+1}(x) = \frac{\sigma^2}{2}\frac{\partial^2 F_n(x)}{\partial x^2} - \Lambda(x)F_n(x) \text{ for all } x \in \mathbb{R} \text{ and } n = 1, 2, \dots \text{ and}$$
$$F_1(x) = -\Lambda(x) \text{ for all } x \in \mathbb{R}$$

Lemma 4 is the base of an algorithm to compute the derivatives of S at $t = 0$ given Λ and the derivatives of Λ at $x = 0$ given S. Using this lemma, we obtain the main result of this section:

Proposition 15. Assume that $\sigma^2 > 0$, $X = \infty$, and Λ satisfies Assumption 1. Let S be the Survival function of Λ, as in equation (76). If the generalized hazard function Λ is analytical, then the Survival function S uniquely identifies Λ. Likewise, if the Survival function S is analytical, then the generalized hazard function Λ uniquely identifies S.

As remarked before, Lemma 4 gives an algorithm to recursively compute an expansion of S based on the derivatives of Λ, or an expansion of Λ based on the derivatives of S. An implication of Lemma 4 and Proposition 15 is that the hazard rate and its first three derivatives at zero duration ($t = 0$) are given by particularly simple expressions involving the level and first two even derivatives of the generalized hazard function evaluated at zero price gap, i.e. $x = 0$:

$$h(0) = \Lambda(0) \geq 0\,,\ \frac{\partial h(t)}{\partial t}|_{t=0} = \frac{\sigma^2}{2}\frac{\partial^2 \Lambda(x)}{\partial x^2}|_{x=0}\,,\ \frac{\partial^2 h(t)}{\partial t^2}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^2 \frac{\partial^4 \Lambda(x)}{\partial x^4}|_{x=0}\,,$$
$$\text{and } \frac{\partial^3 h(t)}{\partial t^3}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^3 \frac{\partial^6 \Lambda(x)}{\partial x^6}|_{x=0} - 4\left(\frac{\sigma^2}{2}\frac{\partial^2 \Lambda(x)}{\partial x^2}|_{x=0}\right)^2$$

These formulas give a simple connection between the local behavior of Λ around $x = 0$ and h around $t = 0$. Note that if $\Lambda(x)$ is, in addition of being symmetric and differentiable in x, increasing in $|x|$ around $x = 0$, then $\Lambda''(0) > 0$, and hence the hazard rate as function of duration, $h(t)$, must be increasing in duration, at least for small durations t. Likewise, if $\Lambda(x)$ were decreasing in $|x|$ around $x = 0$, then $\Lambda''(0) < 0$ and hence $h(t)$ must be locally decreasing in duration.

Comparing with the case of Theorem 2, in this case we use much more restrictive conditions for Λ, and obtain a more cumbersome representation — an infinite expansion instead of a closed-form expression involving an integral. In spite of this Theorem 2 and Proposition 15 have the same flavor: they show that if Λ is analytical and $X = \infty$, then Λ can be fully identified either using the information contained in the Survival function, i.e duration on price changes, and σ^2, which can be recovered from N_a and the variance of price changes with equation (13). Of course, this also means that the information on the survival function and the size distribution of price changes can be used as an over-identifying test of the model.

Finally, we can also estimate $\mathcal{C} \equiv \Lambda(0)/N_a$, the fraction of price changes independent of the state, by using duration data. Given the results above, $\mathcal{C}$ can be estimated as $h(0)/N_a$. This can be an alternative to the estimates presented in Table 1 using the size distribution of price changes. As in Section 4, a correction of unobserved heterogeneity may be important.

F Discrete Distribution of Fixed Costs

Let $g_i > 0$ be the probability of drawing a fixed cost ψ_i for $i = 1, \ldots, n-1$, conditional of drawing a low adjustment cost opportunity. We have $0 < \psi_1 < \cdots < \psi_{n-1}$. A firm can always pay a fixed cost $\Psi \equiv \psi_n$ and change prices, with $\psi_n > \psi_{n-1}$. At all points x where v is twice differentiable we have:

$$rv(x) = \min\left\{ Bx^2 + \frac{\sigma^2}{2} v''(x) \,+\, \kappa \sum_{j=1}^{n-1} \min\left\{\psi_j + v(0) - v(x)\,,\, 0\right\} g_j \,,\, r\left(\psi_n + v(0)\right) \right\}$$

The optimal decision rule can be described by n thresholds $0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_n \equiv X$. The optimal decision rule is that conditional on drawing the adjustment cost ψ_j an adjustment takes place if $|x| \geq \bar{x}_j$ for $j = 1, \ldots, n$. Note that this implies that:

$$v(\bar{x}_j) + \psi_j = v(0) \text{ for } j = 1, 2, \ldots, n.$$

To simplify the notation we let:

$$\lambda_j \equiv \kappa g_j \text{ for } j = 1, \ldots, n-1 \text{ and } \Lambda(x) = \sum_{k=1}^{n-1} \lambda_k \, 1_{\{x \geq \bar{x}_k\}}$$

To summarize the firm's problem is defined by parameters $r, B, \sigma^2, \{\lambda_j\}_{j=1}^{n-1}, \{\psi_j\}_{j=1}^{n}$. The solution is given by a set of thresholds $\{\bar{x}_j\}_{j=1}^{n}$ with $0 < \bar{x}_1 < \cdots < \bar{x}_n$.

We can write the value function for each segment $j = 1, 2, \ldots, n$:

$$\left(r + \sum_{k=1}^{j-1} \lambda_k \right) v_j(x) = Bx^2 + \frac{\sigma^2}{2} v_j''(x) \,+\, \sum_{k=1}^{j-1} \left[v_1(0) + \psi_k\right] \lambda_k \text{ for } x \in (\bar{x}_{j-1}\,,\, \bar{x}_j]$$

where for convenience we define $\bar{x}_0 = 0$. The value function v must be differentiable at all $x \in \mathbb{R}$, and twice differentiable for all $x \in \mathbb{R}$, except $x = \bar{x}_j$ for $j = 1, \ldots, n$. Thus we have the boundary conditions:

$$v'(0) = v'(\bar{x}_n) = 0$$

F.1 Value function for discrete ψ distribution

The solution of the value function v is characterized by coefficients $\{a_j, b_j, c_j\}_{j=1}^{n}$, roots $\{\eta_j\}_{j=1}^{n}$ and thresholds $\{\bar{x}_j\}_{j=1}^{n}$. In particular, given the thresholds $\{\bar{x}_j\}_{j=1}^{n}$ we write a linear o.d.e. for each segment $[\bar{x}_{j-1}, \bar{x}_j]$ for $j = 1, \ldots, n$. This o.d.e. is parametrized by three constants a_j, b_j, c_j as follows:

$$v_j(x) = a_j + b_j x^2 + c_j \left(e^{\eta_j x} + e^{-\eta_j x}\right) \text{ for } x \in [\bar{x}_{j-1}, \bar{x}_j] \text{ and } j = 1, \ldots, n$$

where η_j is given by:

$$\eta_j = \sqrt{\frac{(r + \sum_{k=1}^{j-1} \lambda_j)}{\sigma^2/2}}$$

Replacing the non-homogenous solution $a_j + b_j x^2$ into the o.d.e. in each segment we have:

$$\left(r + \sum_{k=1}^{j-1} \lambda_k\right)(a_j + b_j x^2) = Bx^2 + \frac{\sigma^2}{2} 2b_j + \sum_{k=1}^{j-1} [v_1(0) + \psi_k]\, \lambda_k \text{ for } x \in [\bar{x}_{j-1}, \bar{x}_j] \text{ and } j = 1, \ldots, n$$

Matching the terms quadratic in x, and using that $v_1(0) = a_1 + 2c_1$, we get:

$$\left(r + \sum_{k=1}^{j-1} \lambda_k\right) b_j = B \quad \text{for } j = 1, \ldots, n \tag{78}$$

Matching the constant we have:

$$\left(r + \sum_{k=1}^{j-1} \lambda_k\right) a_j = \sigma^2 b_j + \sum_{k=1}^{j-1} [a_1 + 2c_1 + \psi_k]\, \lambda_k \quad \text{for } j = 1, \ldots, n \tag{79}$$

The continuity and (once) differentiability at $x = \bar{x}_j$ for $j = 1, \ldots, n-1$ gives:

$$a_{j+1} + b_{j+1} (\bar{x}_j)^2 + c_{j+1} \left(e^{\eta_{j+1} \bar{x}_j} + e^{-\eta_{j+1} \bar{x}_j}\right) = a_j + b_j (\bar{x}_j)^2 + c_j \left(e^{\eta_j \bar{x}_j} + e^{-\eta_j \bar{x}_j}\right) \quad \text{for } j = 1, \ldots, n-1 \tag{80}$$

and

$$2b_{j+1}\bar{x}_j + c_{j+1}\eta_{j+1} \left(e^{\eta_{j+1} \bar{x}_j} - e^{-\eta_{j+1} \bar{x}_j}\right) = 2b_j \bar{x}_j + c_j \eta_j \left(e^{\eta_j \bar{x}_j} - e^{-\eta_j \bar{x}_j}\right) \text{ for } j = 1, \ldots, n-1 \tag{81}$$

value matching and smooth pasting at $\bar{x}_n$ gives:

$$\psi_n + a_1 + 2c_1 = a_n + b_n (\bar{x}_n)^2 + c_n \left(e^{\eta_n \bar{x}_n} + e^{-\eta_n \bar{x}_n}\right) \tag{82}$$

$$0 = 2b_n \bar{x}_n + c_n \eta_n \left(e^{\eta_n \bar{x}_n} - e^{-\eta_n \bar{x}_n}\right) \tag{83}$$

The optimal return point conditions, $v'(0) = 0$, is automatically satisfied by symmetry of the value functions.

Thus we have $4 \times n$ unknowns, namely $\{\bar{x}_j, a_j, b_j, c_j\}_{j=1}^n$, and $4 \times n$ equations, namely n equations matching quadratic terms, i.e. equations (78), n equations matching constants, i.e. equations (79), $n-1$ equations enforcing continuity, i.e. equations (80), $n-1$ equations enforcing differentiability, i.e. equations (81), and two more equations on the boundary $\bar{x}_n$ enforcing value matching, i.e. equation (82), and smooth pasting, i.e. equation (83).

F.2 Inverse problem: recovering the cost function

We now solve an inverse problem, namely how to recover the menu cost values ψ_j that underlie a given observed hazard function $\Lambda(x)$ at given thresholds $\{\bar{x}_j\}$. The main result is summarized by the next proposition:

Proposition 16. *Fix a discount rate, curvature and variance $r, B, \sigma^2 > 0$, and a step function*

Λ giving the probability per unit of time of a price adjustment for $|x| < x_n$. The function Λ is described by a set of probability rates for costs $\{\lambda_j\}_{j=1}^{n-1} \in \mathbb{R}_+^n$ for $n \geq 1$, and a set of n thresholds $\{\bar{x}_j\}_{j=1}^n$ with $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_n$. Then there is a unique set of n fixed costs $0 = \psi_0 < \psi_1 < \cdots < \psi_n$ so that the n thresholds $\{\bar{x}_j\}_{j=1}^n$ solve the firm's problem defined by $r, B, \sigma^2, \{\lambda_j\}_{j=0}^{n-1}, \{\psi_j\}_{j=1}^n$. Moreover, the fixed costs $\{\psi_j\}_{j=1}^n$ and the coefficients of the value function $\{a_j, b_j, c_j\}_{j=1}^n$ solve a system of linear equations.

Proof. (of Proposition 16) We first solve for each of the coefficients b_j using equation (78) for each $j = 1, \ldots, n$.

We note that the thresholds $\{\bar{x}_j\}_{j=1}^n$ are given and that roots $\{\eta_j\}_{j=1}^n$ can be computed as functions of given parameters.

Using the coefficients $\{b_j\}_{j=1}^n$, we solve for the coefficients $\{c_j\}_{j=1}^n$. First we solve for c_n enforcing smooth pasting at $\bar{x}_n$ given by equation (83). Using c_n we recursively use c_{j+1} to solve for c_j imposing differentiability between adjecent segments, i.e. equations (81) for $j = n-1, n-2, \ldots, 1$.

Next we solve for the $\{a_j\}_{j=1}^n$, given $\{b_j, c_j\}_{j=1}^n$. First, use $rv(0) = \frac{\sigma^2}{2} v''(0) = \frac{\sigma^2}{2} \left(2b_1 + (\eta_1)^2 2c_1\right)$ and $v(0) = a_1 + 2c_1$ to solve for a_1, namely $a_1 = \frac{\sigma^2}{r}\left(b_1 + \eta_1^2 c_1\right) - 2c_1$. Next, use equations (80) to solve recursively for $\{a_j\}_{j=2}^n$.

Finally, we solve for the fixed costs $\{\psi_j\}_{j=1}^n$ using value matching and the values of $\{a_j, b_j, c_j\}_{j=1}^n$. They give:

$$\psi_j = v(\bar{x}_j) - v(0) = a_j + b_j(\bar{x}_j)^2 + c_j\left(e^{\eta_j \bar{x}_j} + e^{-\eta_j \bar{x}_j}\right) - a_1 - 2c_1$$

for $j = 1, \ldots, n$. $\square$

G Solution for the firm's alternative setup of Section 2.2

The first order condition for choice of ℓ in equation (9) are:

$$c'_-\left(\ell^*(x)\right) \leq v(x) - v(0) \leq c'_+\left(\ell^*(x)\right) \text{ for all } x$$

where $\ell^*(x)$ denotes the optimal policy, and where $c'_-(\cdot)$ and $c'_+(\cdot)$ denote the right and left derivatives of c. As in the previous case, we have that if $\Psi < \infty$ there is a barrier $X < \infty$ for which: $v(X) = v(0) + \Psi$ and $v'(X) = 0$. Finally, by the same reasons as before, we have symmetry, i.e. $v(x) = v(-x)$, and $\ell^*(x) = \ell^*(-x)$. As before we can summarize the decision rule of the firm for $x \in (-X, X)$ with a generalized hazard function:

$$\Lambda(x) = \ell^*(x) \text{ for all } x$$

To simplify the discussion, next we describe the case of a cost c that is continuously differentiable and strictly convex, where we simply have:

$$c'\left(\ell^*(x)\right) = v(x) - v(0) \text{ and } \Lambda(x) = \left(c'\right)^{-1}\left(v(x) - v(0)\right) \text{ for all } x$$

We note that since $v(x)$ is strictly increasing in x for $x \in (0, X)$, and $c(\ell)$ is convex, then $\Lambda(x)$ must also be increasing in x for $x \in (0, X)$.

Replacing ℓ^* into the HBJ equation we obtain:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2} v''(x) + \ell^*(x)\left(v(0) - v(x)\right) + c(\ell^*(x)),\, r\left(\Psi + v(0)\right)\right\}$$

Let us assume that the cost function c has a continuous derivative. Defining, as before $U(x) = v(x) - v(0)$, with $u = U' = v'$, we can differentiate the HBJ equation in $x \in (0, X)$, and use the envelope to obtain:

$$[r + \Lambda(x)]\, u(x) = 2Bx + \frac{\sigma^2}{2} u''(x) \; .$$

Using the boundaries $u(0) = u(X) = 0$, and the logic used in the proof of Theorem 1 it is then straightforward to solve for $\Psi = \int_0^X u(z)dz$. The marginal cost of switching intensity is recovered using $\Lambda(x) = (c')^{-1}(U(x))$. The cost function itself, just as the value function, is only detemined up to an additive constant, which is straightforward to verify from equation (9).

H Properties of Distribution of Menu Cost

In this appendix we note that the posited behavior of Λ in a neighbourhood of $x = 0$ or $x = |X|$ determines whether the underlying density g is bounded. It is shown in equation (2) that the hazard function inherits the shape of the value function because of the underlying optimization: when the firm draws a fixed cost, what matters is how the value of the draw compares to the gains from adjustment. Taking a first order derivative of equation (2) gives

$$\Lambda'(x) = \kappa \; g(v(x) - v(0)) \; v'(x) \tag{84}$$

A bounded density g would make $\Lambda'(x)$ have zero limits at $x = 0$ and $x = X$ because of the smooth-pasting conditions on $v(x)$ at these points. Thus, if the hazard function of the inverse problem (the one that solves for g given Λ) is not flat at 0 or Ψ, then the density g must be diverging. We formalize this observation next:

Corollary 5. Let $\varepsilon > 0$ and suppose $\Lambda'(x)$ is bounded away from zero for $x \in (0, \varepsilon)$. Then $g(\psi)$ is unbounded on any $(0, \psi)$. Likewise, if $\Lambda'(x)$ is bounded away from zero for $x \in (X - \varepsilon, X)$ then $g(\psi)$ is unbounded on any (ψ, Ψ).[27]

We can also characterize the behavior of the density g around $\psi = 0$ for different forms of Λ arounf $x = 0$. Take the limiting elasticity of the hazard

$$\nu = \lim_{x \downarrow 0} \frac{x\Lambda'(x)}{\Lambda(x) - \Lambda(0)}$$

If Λ is symmetric and smooth, it admits a quadratic approximation close to zero, and $\nu = 2$. Interestingly, deviations from $\nu = 2$ imply irregular behavior of g. Theorem 1 states that

$$g(x) = \frac{\Lambda'(x)}{\kappa u(x)}$$

But $u(x)$ converges to zero as $x \to 0$, so the limit is tricky. To resolve the indeterminacy, notice that $u(x)$ goes to zero linearly, since $u''(0) = 0$ (immediate from the equation (4) defining $u(x)$ in Lemma 1). Thus whether the limit is (i) zero, (ii) positive and finite, or (iii) infinite, depends respectively on whether $\Lambda'(x)$ goes to zero (i) faster than a linear rate ($\nu > 2$), (ii) at a linear rate ($\nu = 2$), (iii) slower than a linear rate ($\nu < 2$). We can formalize this:

[27]Since $\Lambda(x)$ is symmetric, to be smooth at zero it has to have $\Lambda'(0) = 0$. The proof is done by standard analysis.

Corollary 6. Suppose that $\Lambda'(x)$ and $g(\psi)$ both have (possibly infinite) right limits at zero. Then $\lim_{\psi \downarrow 0} g(\psi) = \infty$ for $\nu < 2$, $0 < \lim_{\psi \downarrow 0} g(\psi) < \infty$ for $\nu = 2$, and $\lim_{\psi \downarrow 0} g(\psi) = 0$ for $\nu > 2$.

This corollary states that a quadratic hazard function implies a density of ψ that is positive and finite around $\psi = 0$. If the leading term in $\Lambda(x)$ is higher than quadratic ($\nu > 2$) then the density must be zero, meaning that G is flat close to $\psi = 0$. A hazard function with a leading term $\nu < 2$ implies a distribution of ψ with density that is diverging around $\psi = 0$.

I Alternative Normalization

We consider an alternative normalization to one used in Proposition 2. This normalization requires that $X < \infty$. For a triplet $\{\sigma^2, X, \Lambda\}$ we can define a new problem represented by pair $\{\rho, \hat{\Lambda}\}$ where $\hat{\Lambda} : (-1, 1) \to \mathbb{R}_+$ and where ρ is a scalar defined as follows:

$$\hat{\Lambda}(z) = \frac{\Lambda(zX)}{\kappa} \text{ for all } z \in [-1, 1] \text{ and } \rho = \frac{2\kappa X^2}{\sigma^2} \tag{85}$$

Note that this is the normalization used in Proposition 2 with $b = 1/X$. This is a slight generalization of Proposition 2, in that it allows to have some comparative static with respect to κ.

Given the triplet $\{\sigma^2, X, \Lambda\}$ we can solve for f as indicated in equation (10). And given the pair $\{\rho, \hat{\Lambda}\}$ we can solve for the probability density $\hat{f}$, using a change of variables:

$$\hat{f}(z) \equiv f(zX)\, X \text{ for all } z \in [-1, 1]$$

We note that $\hat{f}$ satisfies the

$$\hat{\Lambda}(z)\, \rho\, \hat{f}(z) = \hat{f}''(z) \text{ for all } z \in [-1, 1] \text{ and } z \notin \mathbb{Z} \tag{86}$$

where $z \in \mathbb{Z}$ if $z = x/X$ and $x \in \mathbb{J}$. Moreover, the density $\hat{f}$ must satisfy

$$\hat{f}(1) = \hat{f}(-1) = 0 \text{ and } \int_{-1}^{1} \hat{f}(z) dz = 1$$

J Functional forms of $\langle f(x), m(x), \mathcal{T}(x) \rangle$ for integer ν

The invariant density f has to be symmetric around $x = 0$, and has to satisfy:

$$\Lambda(x) f(x) = \frac{\sigma^2}{2} f''(x) \text{ for all } x \in [0, X], \tag{87}$$

$$\frac{1}{2} = \int_0^X f(x) dx \text{ and } f(X) = 0. \tag{88}$$

The contribution of an individual firm to the IRF is antisymmetric around $x = 0$ and satisfies the following:

$$\Lambda(x)m(x) = -x + \frac{\sigma^2}{2}m''(x) \text{ for all } x \in [0, X]\,, \tag{89}$$

$$m(0) = m(X) = 0\,. \tag{90}$$

Fianlly, $\mathcal{T}(x)$ is symmetric around $x = 0$ and satisfies

$$\Lambda(x)\mathcal{T}(x) = 1 + \frac{\sigma^2}{2}\mathcal{T}(x) \text{ for all } x \in [0, X]\,,$$
$$\mathcal{T}(X) = 0 \text{ and } \mathcal{T}'(0) = 0\,.$$

The latter equality is a consequence of $\mathcal{T}(\cdot)$ being continuously differentiable ay zero and antisymmetric.

Denote $y = \sigma^2/2a$. We will assume that the functions of interest are analytical, so we can write them as:

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ for } x \in [0, X]$$
$$m(x) = \sum_{k=0}^{\infty} \beta_k x^k \text{ for } x \in [0, X]$$
$$\mathcal{T}(x) = \sum_{k=0}^{\infty} \gamma_k x^k \text{ for } x \in [0, X]$$

so that, in particular, $\gamma_0 = \mathcal{T}(0)$. Inserting these expressions into the equations above and using the functional form for $\Lambda(\cdot)$, we obtain:

$$a\sum_{k=0}^{\infty} \alpha_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \alpha_k k(k-1)x^{k-2} \qquad \text{for } x \in [0, X]$$
$$a\sum_{k=0}^{\infty} \beta_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \beta_k k(k-1)x^{k-2} - x \qquad \text{for } x \in [0, X]$$
$$a\sum_{k=0}^{\infty} \gamma_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \gamma_k k(k-1)x^{k-2} + 1 \qquad \text{for } x \in [0, X]$$

Matching the coefficient of each of the powers of x we have

$$\alpha_k = y(k+\nu+2)(k+\nu+1)\alpha_{k+\nu+2} \text{ for } k \geq 0$$
$$\beta_k = y(k+\nu+2)(k+\nu+1)\beta_{k+\nu+2} \text{ for } k \geq 0$$
$$\gamma_k = y(k+\nu+2)(k+\nu+1)\gamma_{k+\nu+2} \text{ for } k \geq 0$$

The symmetry and smoothness properties also lead to

$$\beta_0 = \beta_2 = \gamma_1 = 0 \tag{91}$$

Relabelling the coefficients, we can write the sums as

$$
\begin{aligned}
f(x) &= \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} y^{-j} x^{j(\nu+2)}\right) + \alpha_1 x \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} y^{-j} x^{j(\nu+2)}\right) \\
m(x) &= \beta_1 x \left(1 + \sum_{j=1}^{\infty} \xi_{m,j} y^{-j} x^{j(\nu+2)}\right) + \beta_3 x^3 \left(1 + \sum_{j=1}^{\infty} \eta_{m,j} y^{-j} x^{j(\nu+2)}\right) \\
\mathcal{T}(x) &= \gamma_0 \left(1 + \sum_{j=1}^{\infty} \xi_{t,j} y^{-j} x^{j(\nu+2)}\right) + \gamma_2 x^2 \left(1 + \sum_{j=1}^{\infty} \eta_{t,j} y^{-j} x^{j(\nu+2)}\right)
\end{aligned}
$$

Here the coefficients $\xi_{\cdot,j}$ and $\eta_{\cdot,j}$ are given by

$$
\begin{aligned}
\xi_{p,j} &= \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)-1)} & \eta_{p,j} &= \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)+1)} \\
\xi_{m,j} &= \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)+1)} & \eta_{m,j} &= \prod_{i=1}^{j} \frac{1}{(i(\nu+2)+2)(i(\nu+2)+3)} \\
\xi_{t,j} &= \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)-1)} & \eta_{t,j} &= \prod_{i=1}^{j} \frac{1}{(i(\nu+2)+1)(i(\nu+2)+2)}
\end{aligned}
$$

Now define the following parameter:

$$
Z = \frac{X^{\nu+2}}{y} = 2aX^{\nu}\frac{X^2}{\sigma^2} = 2\kappa\mathcal{T}_0
$$

It will be useful in pinning down the coefficients. Here $\tilde{\Lambda}$ is the left limit of the hazard rate when x approaches X, and $\mathcal{T}_0$ is the expected time to adjustment when $a = 0$.

Consider first $f(\cdot)$. The boundary condition is

$$
\begin{aligned}
0 = f(X) &= \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} y^{-j} X^{j(\nu+2)}\right) + \alpha_1 x \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} y^{-j} X^{j(\nu+2)}\right) \\
&= \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} Z^j\right) + \alpha_1 X \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} Z^j\right)
\end{aligned}
$$

Define additionally $\xi_{\cdot,0} = \eta_{\cdot,0} = 1$. The condition that $f(\cdot)$ is a density states

$$
\begin{aligned}
\frac{1}{2} = \int_0^X f(x)dx &= \alpha_0 X \left(1 + \sum_{j=1}^{\infty} \frac{\xi_{p,j} Z^j}{j(\nu+2)+1}\right) + \alpha_1 X^2 \left(\frac{1}{2} + \sum_{j=1}^{\infty} \frac{\eta_{p,j} Z^j}{j(\nu+2)+2}\right) \\
&= \alpha_0 X \sum_{j=0}^{\infty} \frac{\xi_{p,j} Z^j}{j(\nu+2)+1} + \alpha_1 X^2 \sum_{j=0}^{\infty} \frac{\eta_{p,j} Z^j}{j(\nu+2)+2}
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\alpha_1 &= \frac{1}{2X^2}\frac{\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)}{\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)-\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}\\
&= \frac{1}{2X^2}\hat{\alpha}_1(\nu,Z)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_0 &= -\frac{1}{2X}\frac{\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}{\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)-\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}\\
&= \frac{1}{2X}\hat{\alpha}_0(\nu,Z)
\end{aligned}
$$

Now observe that the integral of $f(x)x^2$ is in fact proportional to X^2 for a fixed Z:

$$
\begin{aligned}
\int_0^X f(x)x^2dx &= \alpha_0X^3\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+3}+\alpha_1X^4\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+4}\\
&= \frac{X^2}{2}\left[\hat{\alpha}_0(\nu,Z)\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+3}+\hat{\alpha}_1(n,Z)\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+4}\right]
\end{aligned}
$$

To determine $m(\cdot)$ and $\mathcal{T}(\cdot)$, it is useful to consider separately the cases $\nu \geq 1$ and $\nu = 0$. Start with $\nu \geq 1$. In this case, in addition to equation (91), we know that

$$3\sigma^2\beta_3 = 1 \text{ and } \sigma^2\gamma_2 = -1$$

The boundary conditions are $m(X) = \mathcal{T}(X) = 0$, so

$$
\begin{aligned}
-\beta_1 &= \frac{X^2}{3\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{m,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{m,j}Z^j}\right)\\
\gamma_0 &= \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right)
\end{aligned}
$$

The functional forms are then

$$
\begin{aligned}
m(x) &= -\frac{xX^2}{3\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{m,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{m,j}Z^j}\right)\sum_{j=0}^{\infty}\xi_{m,j}y^{-j}x^{j(\nu+2)}+\frac{x^3}{3\sigma^2}\sum_{j=0}^{\infty}\eta_{m,j}y^{-j}x^{j(\nu+2)}\\
\mathcal{T}(x) &= \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right)\sum_{j=0}^{\infty}\xi_{t,j}y^{-j}x^{j(\nu+2)}-\frac{x^2}{\sigma^2}\sum_{j=0}^{\infty}\eta_{t,j}y^{-j}x^{j(\nu+2)}
\end{aligned}
$$

Observe that for $\mathcal{T}(0)$ we have

$$\mathcal{T}(0) = \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right) = \mathcal{T}_0\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}(2\kappa\mathcal{T}_0)^j}\right)$$

At $a = 0$ or, equivalently, $\kappa = 0$, we have $\mathcal{T}(0) = \mathcal{T}_0$.

Now consider the case $\nu = 0$. Here the conditions we add to equation (91) are

$$a\beta_1 = 3\sigma^2\beta_3 - 1 \text{ and } a\gamma_0 = \sigma^2\gamma_2 + 1 \tag{92}$$

Plugging them into the boundary conditions $m(X) = \mathcal{T}(X) = 0$,

$$\begin{aligned}
-\beta_1 &= \frac{X^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}\\
\beta_3 &= \frac{\sum_{j=0}^{\infty}\xi_{m,j}Z^j}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}\\
\gamma_0 &= \frac{X^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}\\
-\gamma_2 &= \frac{\sum_{j=0}^{\infty}\xi_{t,j}Z^j}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}
\end{aligned}$$

The functional forms in this case are

$$\begin{aligned}
m(x) &= -x\frac{X^2\left(\sum_{j=0}^{\infty}\eta_{m,j}Z^j\right)\left(\sum_{j=0}^{\infty}\xi_{m,j}y^{-j}x^{j(\nu+2)}\right)}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}\\
&\quad + x^3\frac{\left(\sum_{j=0}^{\infty}\xi_{m,j}Z^j\right)\left(\sum_{j=0}^{\infty}\eta_{m,j}y^{-j}x^{j(\nu+2)}\right)}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}\\
\mathcal{T}(x) &= \frac{X^2\left(\sum_{j=0}^{\infty}\eta_{t,j}Z^j\right)\left(\sum_{j=0}^{\infty}\xi_{t,j}y^{-j}x^{j(\nu+2)}\right)}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}\\
&\quad - x^2\frac{\left(\sum_{j=0}^{\infty}\xi_{t,j}Z^j\right)\left(\sum_{j=0}^{\infty}\eta_{t,j}y^{-j}x^{j(\nu+2)}\right)}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}
\end{aligned}$$

Observe that in this case for $\mathcal{T}(0)$ we have

$$\begin{aligned}
\mathcal{T}(0) &= \frac{X^2}{\sigma^2}\left(\frac{\sum_{j=0}^{\infty}\eta_{t,j}Z^j}{\sum_{j=0}^{\infty}\xi_{t,j}Z^j + \frac{aX^2}{\sigma^2}\sum_{j=0}^{\infty}\eta_{t,j}Z^j}\right)\\
&= \mathcal{T}_0\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}{1+\kappa\mathcal{T}_0+\sum_{j=1}^{\infty}\xi_{t,j}(2\kappa\mathcal{T}_0)^j+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}\right)
\end{aligned}$$

When $\kappa = 0$, we have $\mathcal{T}(0) = \mathcal{T}_0$.

We know that the adjustment frequency is given by

$$N_a = \frac{1}{\mathcal{T}(0)}$$

Hence, the adjustment frequency can be represented as a function of κ and $\mathcal{T}_0$. The same is true for the kurtosis of price changes. From equation (17),

$$\begin{aligned}
Kurt(\Delta p) &= \frac{2\left[\int_0^X x^4\Lambda(x)f(x)dx - X^4\frac{\sigma^2}{2}f'(X)\right]}{N_a}\frac{1}{[Var(\Delta p)]^2} \\
&= \frac{2N_a\left[\int_0^X x^4\Lambda(x)f(x)dx - X^4\frac{\sigma^2}{2}f'(X)\right]}{\sigma^4} = \frac{12N_a}{\sigma^2}\int_0^X f(x)x^2dx \\
&= 6N_a\frac{X^2}{\sigma^2}\frac{\sum_{j=0}^\infty \frac{\eta_{p,j}Z^j}{j(\nu+2)+4}\left(\sum_{j=0}^\infty \xi_{p,j}Z^j\right) - \sum_{j=0}^\infty \frac{\xi_{p,j}Z^j}{j(\nu+2)+3}\left(\sum_{j=0}^\infty \eta_{p,j}Z^j\right)}{\sum_{j=0}^\infty \frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^\infty \xi_{p,j}Z^j\right) - \sum_{j=0}^\infty \frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^\infty \eta_{p,j}Z^j\right)} \\
&= 6N_a\mathcal{T}_0\frac{\sum_{j=0}^\infty \varphi_{K,j}(2\kappa\mathcal{T}_0)^j}{\sum_{j=0}^\infty \chi_{K,j}(2\kappa\mathcal{T}_0)^j}
\end{aligned}$$

Here the coefficients $\{\varphi_{K,j}, \chi_{K,j}\}_{j=0}^\infty$ are given by

$$\begin{aligned}
\varphi_{K,j} &= \sum_{i=0}^j\left(\frac{\xi_{p,j-i}\eta_{p,i}}{i(\nu+2)+4} - \frac{\eta_{p,j-i}\xi_{p,i}}{i(\nu+2)+3}\right) \\
\chi_{K,j} &= \sum_{i=0}^j\left(\frac{\xi_{p,j-i}\eta_{p,i}}{i(\nu+2)+2} - \frac{\eta_{p,j-i}\xi_{p,i}}{i(\nu+2)+1}\right)
\end{aligned}$$

As expected, when $\kappa = 0$ we have $N_a = 1/\mathcal{T}_0$ and

$$Kurt(\Delta p) = 1.$$

The coefficients $\{\varphi_{N,j}, \chi_{N,j}\}$ for N_a are taken from the corresponding formula for $\mathcal{T}(0)$ in the cases $\nu = 0$ and $\nu \geq 1$. In both cases $\varphi_{N,0} = \chi_{N,0} = 1$. To verify $\varphi_{K,0} = -1/12$ and $\psi_{K,0} = 1/2$, plug $\xi_{p,0} = \eta_{p,0} = 1$. For $\varphi_{K,1}$ and $\chi_{K,1}$, recall that

$$\xi_{p,1} = \frac{1}{(\nu+2)(\nu+1)} \text{ and } \eta_{p,1} = \frac{1}{(\nu+2)(\nu+3)}$$

The first derivative of $Kurt(\Delta p)/(6N_a)$ evaluated at $\kappa = 0$ is

$$\frac{\partial}{\partial\kappa}\left(\frac{Kurt(\Delta p)}{6N_a}\right)\Big|_{\kappa=0} = \mathcal{T}_0\frac{\chi_{K,0}\varphi_{K,1} - \varphi_{K,0}\chi_{K,1}}{\chi_{K,0}^2} = -C(6\varphi_{K,1} - \chi_{K,1})$$

for some positive constant C. Plugging the terms,

$$\varphi_{K,1} = -\frac{1}{12(\nu+5)(\nu+6)}$$
$$\chi_{K,1} = -\frac{1}{2(\nu+3)(\nu+4)}$$

Hence,

$$\frac{\partial}{\partial \kappa}\left(\frac{Kurt(\Delta p)}{6N_a}\right)\Big|_{\kappa=0} = \frac{C}{2}\left(\frac{1}{(\nu+5)(\nu+6)} - \frac{1}{(\nu+3)(\nu+4)}\right) < 0$$

This proves the fact that $Kurt(\Delta p)/(6N_a)$ decreases for small κ.

K Special Cases of Interest

K.1 m and f in the discrete, unbounded case

We assume that we can divide $[0,\infty)$ into N segments, each one where $\Lambda(x)$ is constant at the value $\rho_k > 0$ and with thresholds $\{\bar{x}_k\}_{k=0}^N$ as follows. The values of $\{\bar{x}_k\}$ and $\{\rho_k\}$ are given. We let

$$0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{N-1} < \bar{x}_N = \infty$$

The function $\Lambda(x)$ takes N different strictly positive values denoted by $\{\rho_k\}_{k=1}^N$, so that:

$$\Lambda(x) = \rho_k \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k) \text{ for } k = 1, 2, \ldots, N$$
$$0 < \rho_1 < \rho_2 < \cdots < \rho_N .$$

Since $m(\cdot)$ and $f(\cdot)$ solve Kolmogorov equations (backward for $m(\cdot)$ and forward for $f(\cdot)$), on each segment they can parametrized by a pair of unknown constants:

$$m(x) = M_k(x) = -\frac{x}{\rho_k} + u_k e^{\eta_k x} + v_k e^{-\eta_k x} \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k]$$
$$f(x) = \bar{P}_k(x) = p_k e^{\eta_k x} + q_k e^{-\eta_k x} \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k]$$
$$\eta_k = \sqrt{\frac{2\rho_k}{\sigma^2}}$$

for $k = 1, 2, \ldots, N$. We require that $f(\cdot)$ and $m(\cdot)$ be continuously differentiable on $(0,\infty)$. This implies that

$$M_k(\bar{x}_k) = M_{k+1}(\bar{x}_k) \text{ and } M_k'(\bar{x}_k) = M_{k+1}'(\bar{x}_k) \text{ for all } k = 1, 2, \ldots, N-1 \quad (93)$$
$$\bar{P}_k(\bar{x}_k) = \bar{P}_{k+1}(\bar{x}_k) \text{ and } \bar{P}_k'(\bar{x}_k) = \bar{P}_{k+1}'(\bar{x}_k) \text{ for all } k = 1, 2, \ldots, N-1 \quad (94)$$

In addition we have the following conditions. Since m is antisymmetric around zero we require $m(0) = 0$. Since f is a density, it must integrate to one, and since it symmetric it must integrate to one half over positive x. Finally, both m and f should converge to $-x/\rho_N$ and 0 as $x \to \infty$.

These conditions are sometimes referred as no-bubble conditions. Hence:

$$M_1(0) = 0, \ \frac{1}{2} = \int_0^\infty f(x)dx = \sum_{k=1}^N \int_{\bar{x}_{k-1}}^{\bar{x}_k} \bar{P}_k(x)dx, \text{ and } p_N = u_N = 0$$

Overall, we have $2N$ unknowns, namely $\{u_k, v_k\}_{k=1}^N$, and $2N$ linear equations for $m(\cdot)$, namely $2(N-1)$ from equation (93), that $m(0) = 0$, and the no-bubble condition. Likewise for $f(\cdot)$. We can write these equations and solve for the constants. Once we have them we can evaluate:

$$\int_0^\infty x^2 f(x)dx = \sum_{k=1}^N \int_{\bar{x}_{k-1}}^{\bar{x}_k} x^2 \bar{P}_k(x)dx \text{ and}$$

$$\int_0^\infty m(x)f(x)dx = \sum_{k=1}^N \int_{\bar{x}_{k-1}}^{\bar{x}_k} M_k'(x)\bar{P}_k(x)dx\,.$$

and check if:

$$\sum_{k=1}^N \int_{\bar{x}_{k-1}}^{\bar{x}_k} x^2 \bar{P}_k(x)dx = -\sigma^2 \sum_{k=1}^N \int_{\bar{x}_{k-1}}^{\bar{x}_k} M_k'(x)\bar{P}_k(x)dx\,.$$

Now we will determine the coefficients $\{p_k, q_k\}_{k=1}^N$ and $\{u_k, v_k\}_{k=1}^N$. Start with the ones for $\bar{p}(\cdot)$. Combining the continuity and differentiability conditions, we can write the coefficients recursively for $k = 1, 2...N-1$:

$$p_k = \frac{1}{2}\left(1 + \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_{k+1}-\eta_k)x_k} p_{k+1} + \frac{1}{2}\left(1 - \frac{\eta_{k+1}}{\eta_k}\right) e^{-(\eta_{k+1}+\eta_k)x_k} q_{k+1}$$

$$q_k = \frac{1}{2}\left(1 + \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_k-\eta_{k+1})x_k} q_{k+1} + \frac{1}{2}\left(1 - \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_{k+1}+\eta_k)x_k} p_{k+1}$$

We also have the terminal condition $p_N = 0$ and the normalization (the density must integrate to one half over positives). Observe that the coefficients are in fact linear in q_N, so q_N can easily be found from the normalization. The integral is

$$\frac{1}{2} = \int_0^\infty f(x)dx = \sum_{k=0}^{N-1} p_{k+1} \frac{e^{\eta_{k+1}x_{k+1}} - e^{\eta_{k+1}x_k}}{\eta_{k+1}} - \sum_{k=0}^{N-1} q_{k+1} \frac{e^{-\eta_{k+1}x_{k+1}} - e^{-\eta_{k+1}x_k}}{\eta_{k+1}}$$

We can use linearity: letting $p_k = \hat{p}_k q_N$ and $q_k = \hat{q}_k q_N$ and plugging this into the normalization, we can write

$$\frac{1}{2} = \sum_{k=0}^{N-1} \left(\hat{p}_{k+1} \frac{e^{\eta_{k+1}x_{k+1}} - e^{\eta_{k+1}x_k}}{\eta_{k+1}} - \hat{q}_{k+1} \frac{e^{-\eta_{k+1}x_{k+1}} - e^{-\eta_{k+1}x_k}}{\eta_{k+1}} \right) q_N \tag{95}$$

The numbers $\{\hat{p}_k, \hat{q}_k\}_{k=1}^{N-1}$ are easily obtained from $\{p_k, q_k\}_{k=1}^{N-1}$ computed recursively for some presupposed value of q_N. Knowing them, we can recover the real q_N from equation (95) and recompute the real $\{p_k, q_k\}_{k=1}^{N-1}$.

Now we will determine the coefficients for $m(\cdot)$. The continuity and differentiability conditions lead to the following recursive representation:

$$u_k = \frac{1}{2}\left(1+\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_{k+1}-\eta_k)x_k}u_{k+1} + \frac{1}{2}\left(1-\frac{\eta_{k+1}}{\eta_k}\right)e^{-(\eta_{k+1}+\eta_k)x_k}v_{k+1} + \frac{1}{2}\left(x+\frac{1}{\eta_k}\right)\left(\frac{1}{\rho_k}-\frac{1}{\rho_{k+1}}\right)e^{-\eta_k x_k}$$

$$v_k = \frac{1}{2}\left(1+\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_k-\eta_{k+1})x_k}v_{k+1} + \frac{1}{2}\left(1-\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_{k+1}+\eta_k)x_k}u_{k+1} + \frac{1}{2}\left(x-\frac{1}{\eta_k}\right)\left(\frac{1}{\rho_k}-\frac{1}{\rho_{k+1}}\right)e^{-\eta_k x_k}$$

We also have the terminal condition $u_N = 0$ and the antisymmetry condition $m(0) = 0$. The latter one reduces to $u_1 + v_1 = 0$. Now we can observe that all u_k and v_k are in fact affine in v_N: $u_k = \hat{u}_k v_N + \tilde{u}_k$ and $v_k = \hat{v}_k v_N + \tilde{v}_k$. The condition $m(0) = 0$ can be written as

$$0 = u_1 + v_1 = (\hat{u}_1 + \hat{v}_1)v_N + (\tilde{u}_1 + \tilde{v}_1) \tag{96}$$

The coefficients $\{\hat{u}_k, \hat{v}_k\}_{k=1}^{N-1}$ and $\{\tilde{u}_k, \tilde{v}_k\}_{k=1}^{N-1}$ can be found from $\{u_k, v_k\}_{k=1}^{N-1}$ computed recursively for two different presupposed values of v_N (we need two because the functions are affine, not linear). After that, we can recover the real v_N from equation (96) and recompute the real $\{u_k, v_k\}_{k=1}^{N-1}$.